NO INFINITE DIMENSIONAL \( P \) SPACE ADMITS A
MARKUSCHEVICH BASIS

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Abstract. Theorem. Let \( X \) be a Banach space. If \( X \) is a Grothendieck space and \( X \) admits a Markuschevich basis then \( X \) is reflexive. This theorem is used to prove the conjecture of J. A. Dyer [1] stated in the title.

Recall that a Banach space \( X \) is a Grothendieck space if every weak* convergent sequence in \( X^* \) is weakly convergent. \( X \) is a \( P \) space if \( X \) is complemented in every Banach space which contains it as a subspace.\(^1\) Since a complemented subspace of a Grothendieck space is a Grothendieck space and since every \( P \) space can be embedded in the Grothendieck space \( m(T) \) for a suitable set \( T \), every \( P \) space is a Grothendieck space. Infinite dimensional \( P \) spaces are nonreflexive, so Dyer's conjecture is a consequence of our theorem.

Proof of the Theorem. Suppose that \( \{x_i, f_i\}_{i \in I} \) is a Markuschevich basis for the Grothendieck space \( X \); i.e., \( \{x_i, f_i\}_{i \in I} \) is a biorthogonal collection in \( (X, X^*) \) such that \( \{x_i\}_{i \in I} \) is fundamental in \( X \) and \( \{f_i\}_{i \in I} \) is total over \( X \). Let \( Y \) be the norm closure in \( X^* \) of the linear span of \( \{f_i\}_{i \in I} \) and let \( B \) be the closed unit ball of \( Y \).

To show that \( X \) is reflexive it is sufficient to show that \( Y \) is reflexive. (Indeed, \( Y \) is total over \( X \) so that \( Y \) is weak* dense in \( X^* \). If \( B \) is weakly compact,\(^2\) then \( B \) is weak* compact, so that it follows from the Krein-Smulian theorem that \( Y \) is weak* closed and hence \( Y = X^* \).) By Eberlein's theorem, we need to show only that \( B \) is weakly sequentially compact.

Let \( \{y_n\}_{n=1}^\infty \) be a sequence in \( B \). Since each \( y_n \) is the norm limit of a sequence from the linear span of \( \{f_i\}_{i \in I} \), it follows that for each \( n \), the set \( A_n = \{i \in I : y_n(x_i) \neq 0\} \) is countable and thus \( U_{n=1}^\infty A_n \) is countable. A standard diagonalization argument shows that there is an increasing sequence \( \{P(n)\}_{n=1}^\infty \) of positive integers such that \( \lim_{n \to \infty} y_{P(n)}(x_i) \) exists for each \( i \in I \). Since \( \{y_n\}_{n=1}^\infty \) is equicontinuous

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\(^1\) For the basic facts concerning \( P \) spaces see [3]. The most interesting nonreflexive Grothendieck spaces are discussed in [2].

\(^2\) Since the weak topology on \( Y \) by \( Y^* \) is the relativisation to \( Y \) of the weak topology on \( X^* \) by \( X^{**} \), there is no ambiguity in discussing the weak topology on \( Y \).
on $X$ and $\{x_i\}_{i \in I}$ is fundamental in $X$, $\lim_{n \to \infty} y_{P(n)}(x)$ exists for each $x \in X$. That is, $\{y_{P(n)}\}_{n=1}^\infty$ is weak* convergent to, say, $y$ in $X^*$. Since $X$ is a Grothendieck space, $\{y_{P(n)}\}_{n=1}^\infty$ is weakly convergent to $y$. Finally, $y$ is in $Y$ (and hence in $B$) because the weak and norm closures in $X^*$ of the linear span of $\{f_i\}_{i \in I}$ are the same. Thus $B$ is weakly sequentially compact and the proof is complete.

References


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