ON AN INEQUALITY OF T. J. WILLMORE

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Abstract. Willmore proved that the integral of the square of mean curvature $H$ over a closed surface $M^2$ in $E^3$, $\int_M H^2 dV$, is $\geq 4\pi$, and equal to $4\pi$ when and only when $M^2$ is a sphere in $E^3$. In this paper we give some generalizations of Willmore's result.

Let $x: M^2 \to E^3$ be an immersion of an oriented closed surface $M^2$ into euclidean 3-space $E^3$. In [6], [7], Willmore proved the following inequality for the mean curvature $H(p)$ of $M^2$ in $E^3$:

$$\int_{M^2} H^2(p) dV \geq 4\pi,$$

where the equality holds when and only when $M^2$ is imbedded as a sphere. The main aim of this paper is to give some generalizations of the inequality (1).

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1. Preliminaries. Let $M^n$ be an $n$-dimensional oriented closed manifold with an immersion $x: M^n \to E^{n+N}$. Let $F(M^n)$ and $F(E^{n+N})$ be the bundles of oriented orthonormal frames of $M^n$ and $E^{n+N}$ respectively. Let $B$ be the set of elements $b = (p, e_1, \ldots, e_{n+N})$ such that $(p, e_1, \ldots, e_n) \in F(M^n)$ and $(x(p), e_1, \ldots, e_{n+N}) \in F(E^{n+N})$ whose orientation is coherent with the one of $E^{n+N}$, identifying $e_i$ with $dx(e_i)$, $i = 1, \ldots, n$. Then $B \to M^n$ may be considered as a principal bundle with fibre $SO(n) \times SO(N)$, and $\hat{x}: B \to F(E^{n+N})$ is naturally defined by $\hat{x}(b) = (x(p), e_1, \ldots, e_{n+N})$. Let $B_v$ be the set of normal unit vectors of $M^n$ in $E^{n+N}$, and $B_v \to M^n$ is a sphere bundle whose fibre at $p \in M^n$ is $S^{n-1}_p$. Let $\bar{v}: B_v \to S^{n+N-1}_p$ be the mapping such that $\bar{v}(p, e)$ is the unit vector at the origin of $E^{n+N}$ and parallel to $e$.

The structure equations of $E^{n+N}$ are given by
\[ dx = \sum_A \theta_A e_A, \quad de_A = \sum_B \theta_{AB} e_B, \]

\[ d\theta_A = \sum_B \theta_B \wedge \theta_{BA}, \quad d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}, \quad \theta_{AB} + \theta_{BA} = 0, \]

where \( \theta_A, \theta_{AB} \) are differential 1-forms on \( F(E^{n+N}) \). Let \( \omega_A, \omega_{AB} \) be the induced 1-forms on \( B \) from \( \theta_A, \theta_{AB} \) by the mapping \( \tilde{x} \). Then we have

\[ \omega_r = 0, \quad \omega_{ri} = \sum_j A_{rij} \omega_j, \quad A_{rij} = A_{rji}. \]

From (2) we get

\[ d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \quad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}. \]

For any \((p, e_r) \in B\), we put

\[ I = dx \cdot dx, \quad \Pi_r = de_r \cdot dx. \]

The eigenvalues \( k_1(p, e_r), \ldots, k_n(p, e_r) \) of \( \Pi_r \), relative to \( I \) are called the principal curvatures of \( M^n \) associated with \((p, e_r)\). The \( h \)th mean curvature \( H_h(p, e_r) \) associated with \((p, e_r)\) is defined by the following equation

\[ \det(\delta_{ij} + t A_{rij}) = \sum_{h=0}^n \binom{n}{h} H_h(p, e_r) t^h, \]

where \( \delta_{ij} \) is the Kronecker delta. If there is no danger of confusion, we shall simply denote \( k_i(p, e_r) \) and \( H_i(p, e_r) \) by \( k_i(p) \) (or \( k_i \)) and \( H_i(p) \) (or \( H_i \)) respectively. It is easy to see that

\[ \binom{n}{h} H_h = \sum k_1 \cdots k_h, \quad h = 1, 2, \ldots, n, \]

and \( H_0 = 1 \). Throughout this paper, we simply denote \( H_n(p, e_r) \) by \( K(p, e_r) \). \( K(p, e_r) \) is called the Lipschitz-Killing curvature at \((p, e_r)\).

2. \( \alpha \)th curvature of first and second kinds. Let \( M^2 \) be a surface immersed in \( E^{2+N} \). Let \((p, e_1, e_2, e_3, \ldots, e_{2+N})\) be a local cross section of \( B \rightarrow F(M^2) \) and for any \( e \) in \( S_{p}^{N-1}, \; p \in U \), put \( e = e_{2+N} = \sum_r \xi_r \tilde{e}_r(p) \). Denoting the restriction of \( A_{rij} \) on the image of this local cross section by \( \overline{A}_{rij} \), we may put

\[ A_{2+N} = \sum_r \xi_r A_{rij}. \]
From (3) and (6), we get
\[ K(p, e) = \det \left( \sum_{r} \xi_{r} A_{rj} \right) \]
(8)
\[ = \left( \sum_{r} \xi_{r} A_{111} \right) \left( \sum_{r} \xi_{r} A_{122} \right) - \left( \sum_{r} \xi_{r} A_{112} \right)^2. \]
The right-hand side is a quadratic form of $\xi_1, \cdots, \xi_{2+N}$. Hence, by choosing a suitable cross section, we can write $K(p, e)$ as
\[ K(p, e) = \sum_{r} \lambda_{r-2} \xi_{r}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N. \]
(9)
We call this local cross section of $B \to F(M^2)$, the Frenet cross section in the sense of Ōtsuki, and the frame $(p, e_1, e_2, e_3, \cdots, e_{2+N})$ the Ōtsuki’s frame [5]. We call the curvature $\lambda_a$, the $a$th curvature of the second kind. With respect to this Ōtsuki’s frame the curvatures:
\[ \mu_a(p) = H_1(p, e_{a+2}) \]
are called the $a$th curvature of the first kind. By means of the method of definitions, $\mu_a, \lambda_a$ are defined continuous on the whole manifold $M^2$ and are differentiable on the open subset in which $\lambda_1 > \lambda_2 > \cdots > \lambda_N$.
With respect to the Ōtsuki frame, we have [5]
\[ \omega_1 \wedge \omega_r = \lambda_{r-3} dV, \quad dV = \omega_1 \wedge \omega_2, \quad r = 3, \cdots, 2 + N, \]
(11)
\[ G(p) = \sum_{a=1}^{N} \lambda_a(p), \]
(12)
where $G(p)$ denotes the Gaussian curvature of $M^2$ in $E^{2+N}$, and we also have
\[ H_1(p, e) = \sum_{a=1}^{N} \cos \theta_a \mu_a(p), \quad e = \sum_{r} \cos \theta_{r-2} \bar{e}_r. \]
(13)
As in [2], [3], [5], we know that the forms:
\[ d\sigma = \omega_{2+N,1} \wedge \cdots \wedge \omega_{2+N,1+N}, \quad dV \wedge d\sigma \]
can be regarded as the volume elements of $S_p^{N-1}$ and $B$, respectively.

3. Some generalizations of Willmore’s inequality.

Theorem 1. Let $x: M^2 \to E^{2+N}$ be an immersion of an oriented closed surface $M^2$ into $E^{2+N}$. Then the sum of the squares of the $a$th curvatures
of the first kind satisfies the following inequality:

\[ \int_{M^2} \left( \sum_{\alpha=1}^{N} \lambda_{\alpha}^2 \right) dV \geq 4\pi, \]

where the equality holds when and only when \( M^2 \) is imbedded as a sphere in a 3-dimensional linear subspace of \( E^{2+N} \).

**Proof.** Let \( (p, e_1, e_2, \ldots, e_{2+N}) \) be an Ôtsuki's frame, then by (9) we know that the Lipschitz-Killing curvature \( K(p, e) \) satisfies

\[ \int_{B^*} K(p, e) dV \wedge d\sigma = \int_{B^*} \left( \lambda_1(p) \cos^2 \theta_1 + \cdots + \lambda_N(p) \cos^2 \theta_N \right) dV \wedge d\sigma \]

\[ = \frac{c_{N+1}}{2\pi} \int_{M^2} \left( \sum_{\alpha=1}^{N} \lambda_{\alpha}^2(p) \right) dV = \frac{c_{N+1}}{2\pi} \int_{M^2} G(p) dV. \]

Thus by the Gauss-Bonnet formula, we have

\[ \int_{B^*} K(p, e) dV \wedge d\sigma = (2 - 2g)c_{N+1}, \]

where \( g \) denotes the genus of \( M^2 \) and \( c_{N+1} \) denotes the volume of the unit \((N+1)\)-sphere. On the other hand, by an inequality of Chern-Lashof [3], we have

\[ \int_{B^*} |K(p, e)| dV \wedge d\sigma \geq (2 + 2g)c_{N+1}. \]

Therefore, if we set

\[ V^+ = \{(p, e) \in B^* : K(p, e) \geq 0\}, \quad V^- = \{(p, e) \in B^* : K(p, e) < 0\}, \]

then, by (16) and (17), we get

\[ \int_{V^+} K(p, e) dV \wedge d\sigma \geq 2c_{N+1}. \]

It is easy to see that the equality of (19) holds when and only when

\[ \int_{B^*} |K(p, e)| dV \wedge d\sigma = (2 + 2g)c_{N+1}. \]

Now, we have the following identity:

\[ (k_1 + k_2)^2 = (k_1 - k_2)^2 + 4k_1k_2. \]

Hence we get
\[ \int_{B^+ \setminus \sigma} (H_1(p, e))^2 dV \wedge d\sigma \geq \int_{v' \setminus \sigma} (H_1(p, e))^2 dV \wedge d\sigma \]
\[ \quad \geq \int_{v'} K(p, e) dV \wedge d\sigma. \]

Therefore, by (19), we get

\[ \int_{B^+} (H_1(p, e))^2 dV \wedge d\sigma \geq 2c_{N+1}. \]

On the other hand, by (14) and the following formulas:

\[ \int_{\mathcal{D}_{p-1}} \cos \theta_\alpha \cos \theta_\beta d\sigma = \frac{c_{N+1}}{2}, \quad \text{if } \alpha = \beta, \]
\[ = 0, \quad \text{if } \alpha \neq \beta, \]

we have

\[ \int_{B^+} (H_1(p, e))^2 dV \wedge d\sigma = \sum_{\alpha, \beta=1}^{N} \mu_\alpha(p) \mu_\beta(p) \cos \theta_\alpha \cos \theta_\beta dV \wedge d\sigma \]
\[ = \frac{c_{N+1}}{2\pi} \int_{M^2} \left( \sum_{\alpha} \mu_\alpha^2(p) \right) dV. \]

Hence, by (22) and (23), we get (15). Now, suppose that the equality of (15) holds, then, by (19), (20) and (21), we get

\[ H_1(p, e)^2 = K(p, e), \quad \text{and} \quad k_1(p, e) = k_2(p, e), \]

for all \((p, e)\) in \(B^+\). Thus, by (9) and (24), we know that the last curvature of the second kind \(\lambda_N \geq 0\), for all \(p\) in \(M^2\). Hence, by Lemma 1 of [2], we know that \(M^2\) is imbedded as a convex surface in a 3-dimensional linear subspace, say \(E^8\), of \(E^{2+N}\). Thus, by (24), we know that \(M^2\) is imbedded as a sphere in \(E^8\). Conversely, if \(M^2\) is imbedded as a sphere in a 3-dimensional linear subspace of \(E^{2+N}\), then it is easy to see that the equality of (15) holds. This completes the proof of the theorem.

In the following, we assume that \(x: M^{2m} \to E^{2m+1}\) is an immersion of an oriented even-dimensional closed manifold \(M^{2m}\) into \(E^{2m+1}\). Let \(\varepsilon\) denote the outer normal vector on \(M^{2m}\) in \(E^{2m+1}\). Set

\[ g(p) = H_m(p)^2 - K(p), \]

where \(H_m(p) = H_m(p, \varepsilon)\) and \(K(p) = K(p, \varepsilon)\). By Theorem 1 of [1], we have the following proposition. We omit the proof.

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Proposition 2. Let \( x: M^{2m} \rightarrow E^{2m+1} \) be an immersion of an oriented \( 2m \)-dimensional closed manifold \( M^{2m} \) in \( E^{2m+1} \) with the following property:

\[
\{ p \in M^{2m} : g(p) \geq 0 \} \supseteq \{ p \in M^{2m} : K(p) \geq 0 \}.
\]

Then we have the following inequality:

\[
\int_{M^{2m}} H_m(p)^2 dV \geq \frac{1}{2} c_{2m} \sum_i \beta_i,
\]

where \( \beta_i \) denotes the \( i \)-th betti number of \( M^{2m} \). The equality of (26) holds when and only when the immersion \( x \) is a minimal imbedding [1].

Remark. If \( m = 1 \), then (A) always holds.

Theorem 3. Let \( x: M^{2m} \rightarrow E^{2m+1} \) be an immersion of an oriented \( 2m \)-dimensional closed manifold \( M^{2m} \) in \( E^{2m+1} \) with nonnegative principal curvatures. Then the \( m \)-th mean curvature \( H_m \) satisfies the following:

\[
\int_{M^{2m}} (H_m(p))^2 dV \geq \frac{1}{2} c_{2m} \sum_i \beta_i,
\]

where the equality holds when and only when \( M^{2m} \) is imbedded as a sphere.

Proof. By the assumption, \( k_1, \ldots, k_n \geq 0 \), we have [4, pp. 104–105]

\[
H_m^2 \geq K \geq 0,
\]

where the equality holds when and only when \( k_1 = k_2 = \cdots = k_n \). Thus, by Proposition 2, we have (26). Furthermore, by Theorem 3 of [1], we know that all odd-dimensional betti numbers of \( M^{2m} \) vanish. Hence, by (26), we get (27). Now, suppose that the equality of (27) holds. Then we get \( H_m^2(p) = K(p) \) for all \( p \in M^{2m} \). Hence, we get \( k_1(p) = \cdots = k_n(p) \) for all \( p \in M^{2m} \). Thus \( M^{2m} \) is imbedded as a sphere in \( E^{2m+1} \). This completes the proof of the theorem.

References


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