A CHARACTERIZATION OF TOTAL GRAPHS

M. BEHZAD

I. Abstract. We consider "ordinary" graphs; that is, finite undirected graphs with no loops or multiple edges. The total graph $T(G)$ of a graph $G$ is that graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if they are adjacent or incident in $G$. A characterization of regular total graphs as well as some other properties of total graphs have been considered before. In this article we consider nonregular graphs and yield a method which enables us actually to determine whether or not they are total.

II. Introduction. We consider (ordinary) graphs; that is, finite undirected graphs with no loops or multiple edges. Besides the chromatic number $\chi(G)$ and edge chromatic number $\chi_e(G)$ there is associated with $G$ another positive integer $\chi_t(G)$, called the total chromatic number of $G$, which is the minimum number of colors required for coloring the elements (edges and vertices) of $G$ such that no two elements which are either adjacent or incident have the same color. (The Total Chromatic Conjecture [3] states that $\chi_t(G) \leq 2 + \max \deg G$. In this conjecture $G$ can be replaced by multigraphs $M$ containing no 4-regular multigraph of order 3 as a subgraph.)

The total graph $T(G)$ of $G$ is defined in such a way that $\chi_t(G) = \chi(T(G))$—in analogy with the well-known formula $\chi_e(G) = \chi(L(G))$, where $L(G)$ is the line graph of $G$. The total graph $T(G)$ of $G$ is that graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in $G$.

For an illustration a graph $G$ is given in Figure 1 together with $L(G)$ and $T(G)$. Both $G$ and $L(G)$ are disjoint induced subgraphs of $T(G)$.

A characterization of regular total graphs is given in [7] and some other properties of total graphs are considered in [1], [2], [4]. In this article we yield a method which enables us actually to determine whether or not any given graph is total.

III. Results. In [6] it was proved that $G_1 = G_2$-isomorphism is denoted by the equality sign—if and only if $T(G_1) = T(G_2)$. There, it

Received by the editors January 25, 1970.

AMS 1969 subject classifications. Primary 0540.
Key words and phrases. Total graphs.

1 This restriction on $M$ is necessary as was observed by the author, E. Jacobić and others.

383
was also shown that, apart from isomorphism, \( G \) is the only subgraph of \( H = T(G) \) whose total graph is \( H \), where \( G \) is a connected graph which is neither a cycle \( C \) nor a complete graph \( K \). (Definitions not given here may be found in [5], [8].) This subgraph called the special subgraph of \( H \) is denoted by \( G_s \). The subgraph \( G_s \) is induced by a set of vertices of \( H \) which are called the special vertices of \( H \); any other element of \( V(H) \) is a nonspecial element of \( V(H) \). The subgraph induced by the set of nonspecial vertices of \( H \) is \( L(G_s) \) and in \( H \) each nonspecial vertex is adjacent with exactly two adjacent special elements.

Suppose that \( H \) is a connected total graph which is neither \( T(C) \) nor \( T(K) \), and that \( v \in V(H) \) is nonspecial. Let \( \{i\} \), \( i = 0, 1, 2, \ldots, n \), denote the class of all vertices of \( H \) whose distance from \( v \) is \( i \). Our hypotheses imply that \( n \geq 2 \). Then we have the following theorem.

**Theorem 1.** Let \( H, H \neq T(C), T(K) \), be a connected total graph and let \( v \) be a nonspecial element of \( V(H) \). Then each nonspecial vertex of \( H \) in \( \{i\} \), \( i \geq 1 \), is adjacent with exactly two special vertices of \( H \) both of which are in \( \{i\} \), or one is in \( \{i\} \) and the other in \( \{i+1\} \).

**Proof.** We use induction on \( i \). Let the two adjacent special vertices of \( H \) which are adjacent with \( v \) be denoted by \( u_1 \) and \( v_1 \). It is clear that \( u_1 \) and \( v_1 \) are the only special elements of \( V(H) \) which are in \( \{1\} \); and that each nonspecial element of \( \{1\} \) is adjacent with two special vertices of \( H \) one of which is in \( \{2\} \) and the other in \( \{1\} \).

Let \( w \) be a nonspecial vertex of \( H \) in \( \{2\} \). If \( w \) is adjacent with \( u_1 \) or \( v_1 \), then \( w \) and \( v \) are adjacent in \( H \) and \( w \in \{1\} \) which is a contradiction. Clearly \( w \) is adjacent with no element of \( \{i\}, i \geq 4 \). Next, we show that \( w \) is not adjacent with two special elements of \( \{3\} \). Assume this is the case. Then the vertex \( w \), which is adjacent neither with \( u_1 \) nor with \( v_1 \), is adjacent with a nonspecial element \( w_1 \) of \( \{1\} \). But this is impossible, since the preimages of \( w \) and \( w_1 \) under the total graph function have no vertices in common. Hence the assertion follows for \( \{2\} \).
Assume the assertion is true for \( i, 2 \leq i < n \), and let \( w \) be a non-special vertex of \( H \) in \( \{i+1\} \). First, we show that \( w \) is not adjacent with two special elements of \( \{i+2\} \). Assume this is the case. The vertex \( w \) is adjacent with an element \( w_i \) of \( \{i\} \). The vertex \( w_i \) is not special since otherwise \( w \) is adjacent with three or more special vertices of \( H \). \( w_i \) is not nonspecial either, because otherwise the induction hypothesis would imply that the vertices \( w \) and \( w_i \) are not adjacent.

Next, we show that \( w \) is not adjacent with any special vertex of \( H \) in \( \{i\} \). Assume this is not the case, and let \( u_i \) be a special vertex of \( H \) adjacent with \( w \) which is in \( \{i\} \). The vertex \( u_i \) is adjacent with a vertex \( u \) of \( \{i-1\} \). If \( u \) is special, then by the induction hypothesis the image of the edge \( uu_i \) under the total graph function, say \( u_{i-1} \), is in \( \{i-1\} \), and the vertices \( w \) and \( u_{i-1} \) are adjacent. Thus \( w \in \{i+1\} \) which contradicts our assumption. Hence \( u \) is a nonspecial vertex of \( \{i-1\} \). Then \( u \) is adjacent with a special element of \( \{i-1\} \), say \( u' \). In this case the image of the edge \( uu' \) is \( u \). Hence \( u \) and \( w \) are adjacent and it follows that \( w \in \{i+1\} \). This contradiction proves our assertion and completes the proof of the theorem.

The following corollaries are of value to us.

**Corollary 1.** Under the assumptions of Theorem 1, no nonspecial element of \( \{i\} \) is adjacent with a special element of \( \{i-1\} \), \( i \geq 2 \).

**Corollary 2.** Under the assumptions of Theorem 1, every class \( \{i\}, 1 \leq i \leq n-1 \), contains both special and nonspecial vertices of \( H \); while the class \( \{n\} \) cannot solely contain some nonspecial vertices of \( H \).

**Proof.** Assume \( H = T(G) \). Then \( G \) and \( L(G) \) are two disjoint connected subgraphs of \( H \). Hence if \( \{i\}, 1 \leq i \leq n-1 \), solely contains nonspecial (resp. special) vertices of \( H \), then the class \( \{j\}, j \geq i+1 \), cannot contain any special (resp. nonspecial) element of \( V(H) \). This observation together with Theorem 1 imply that no class \( \{i\}, 1 \leq i \leq n-1 \), solely contains nonspecial (resp. special) vertices of \( H \).
can entirely contain some nonspecial vertices of \( H \). Next, we show that no class \( \{ i \} \), \( 1 \leq i \leq n - 1 \), can contain only special elements. Assume so, and let \( u \in \{ i \} \) for which there exists a special element \( w \) in \( \{ i + 1 \} \) adjacent with \( u \). Then, under the total graph function the edge \( uw \) has no image. This contradiction proves the corollary.

In Figure 2 we present a graph \( H \) which is total and in which the class \( \{ n \} = \{ 3 \} \) contains a special element alone. (We note that \( G_s \) is the path \( v_4, v_5, v_6, v_7 \). Hence the vertices \( v_1, v_2 \) and \( v_3 \) are the nonspecial vertices of \( H \).) The classes \( \{ i \}, i = 0, 1, 2, 3, \) are determined with respect to the vertex \( v_1 \) and they are: \( \{ 0 \} = \{ v_1 \}, \{ 1 \} = \{ v_5, v_6, v_7 \}, \{ 2 \} = \{ v_3, v_6 \}, \) and \( \{ 3 \} = \{ v_4 \} \).

**Corollary 3.** Under the assumptions of Theorem 1, each nonspecial element in \( \{ i \} \) is adjacent with at least one nonspecial element in \( \{ i - 1 \} \), for \( i \geq 1 \), and each special element in \( \{ i \}, i \geq 2 \), is adjacent with at least one special element in \( \{ i - 1 \} \).

**Proof.** This follows directly from Theorem 1, and the fact that both \( G_s \) and \( L(G_s) \) are two connected subgraphs of \( H \).

**Theorem 2.** Assume \( H, H \neq T(C), T(K) \), is a connected total graph and that \( v \) is a nonspecial element of \( V(H) \) adjacent with the special vertices \( v_1 \) and \( u_1 \) of \( H \). Then we can determine the subgraph \( G_s \) completely.

**Proof.** Since \( G_s \) is an induced subgraph of \( H \), it suffices to determine the set of special vertices of \( H \). We do this by separating the sets of special and the sets of nonspecial elements \( S_i \) and \( N_i \), respectively, of each class \( \{ i \}, i = 0, 1, 2, \ldots, n \), formed with respect to the vertex \( v \).

It is clear that the class \( \{ 0 \} \) contains no special element. Thus \( S_0 = \emptyset \), and \( N_0 = \{ 0 \} - S_0 = \{ 0 \} = \{ v \} \). The class \( \{ 1 \} \) contains exactly two special elements and they are the vertices \( u_1 \) and \( v_1 \). Therefore \( S_1 = \{ u_1, v_1 \} \), and \( N_1 = \{ 1 \} - S_1 \). Clearly \( N_1 \neq \emptyset \).

Let \( w_1 \in N_1 \). Then by the proof of Theorem 1, \( w_1 \) is adjacent with one of \( u_1 \) and \( v_1 \), say \( u_1 \), and a special element of \( H \), say \( u_2 \), which is in \( \{ 2 \} \). This vertex \( u_2 \) which we propose to determine is necessarily adjacent with \( u_1 \). Since no nonspecial element of \( \{ 2 \} \) is adjacent with a special element of \( \{ 1 \} \) any element of \( \{ 2 \} \) adjacent with \( u_1 \) is necessarily special. Among these only one is adjacent with both \( u_1 \) and \( w_1 \) since otherwise the vertex \( w_1 \) will be adjacent with three or more special vertices of \( H \). Thus the vertex \( u_2 \) can uniquely be determined. We repeat this argument for all elements of \( N_1 \) and in this manner.
we obtain a set $S'_2$ consisting of some special elements of $\{2\}$. We observe that $S'_2 \neq \emptyset$ since $N_1 \neq \emptyset$. Let $w_2 \in \{2\} - S'_2$. The vertex $w_2$ is a nonspecial element of $H$. For otherwise, by Corollary 3, the vertex $w_2$ is adjacent with a special element, say $v'$, of $\{1\}$ and the edge $w_2v'$ must correspond, under the total graph function, to a nonspecial element, say $w'$, of $H$. But the vertex $w'$ can only be in the class $\{1\}$. This contradicts the fact that $w_2$ is an element of $\{2\} - S'_2$. Thus $N_2 = \{2\} - S'_2$ and $S_2 = S'_2$.

Next, we separate special and nonspecial elements of $\{3\}$. Let $w_2 \in N_2$. If $w_2$ is adjacent with two elements of $S_2$, then each element of $\{3\}$ adjacent with $w_2$ is a nonspecial element of $\{3\}$; otherwise $w_2$ is adjacent with a special element, say $u_2$, of $\{2\}$ and a special element, say $u_3$, of $\{3\}$. We propose to determine $u_3$. Again, as was seen before, all vertices of $\{3\}$ adjacent with $u_2$ are special vertices of $\{3\}$ among which only one is adjacent with both $w_2$ and $u_2$. This vertex is the vertex $u_3$ which we are looking for. We repeat this argument for every element of $N_2$ and obtain a set $S'_3$ consisting of some special elements of $\{3\}$. We observe that every element of $\{3\} - S'_3$ is nonspecial. (The proof is similar to the one given for $\{2\} - S'_2$.) Thus $N_3 = \{3\} - S'_3$, and $S_3 = S'_3$.

Using induction and the above procedure we separate the special and nonspecial elements of $\{i\}$, $i = 0, 1, 2, \ldots, n$. Now we let $S = S_0 \cup S_1 \cup \cdots \cup S_n$. Then $G_* = \langle S \rangle$, and $H = T(G_*)$. This completes the proof of the theorem.

Since in the above theorem the vertices $v$, $u_1$, and $v_1$ play an important role for the determination of $G_*$, we denote $G_*$ by $G_{v; u_1v_1}$.

Let $u$ be an arbitrary vertex of a graph $H$. We denote the set consisting of $u$ and all vertices of $H$ adjacent with $u$ by $\overline{N}(u)$; this set is called the closed neighborhood of $u$. Now we are prepared to present the main theorem of this article.

**Theorem 3.** Assume that $H$, $H \neq T(C)$, $T(K)$, is a connected graph, and that $u$ is an arbitrary vertex of $H$. Then $H$ is total if and only if $H = T(G_{v; u_1v_1})$ for some $v \in \overline{N}(u)$ and some edge $u_1v_1$, where $u_1$ and $v_1$ are two even vertices of $H$ adjacent with $v$.

**Proof.** Assume that $H$ is total. Then some element $v$ of $\overline{N}(u)$ is a nonspecial vertex of $H$. Under the total graph function, $v$ corresponds to an edge $u_1v_1$ of the special subgraph of $H$. The vertices $u_1$ and $v_1$ have even degrees in $H$ and by the definition of total graphs both $u_1$ and $v_1$ are adjacent with $v$. Thus the special subgraph of $H$ is $G_{v; u_1v_1}$ and $H = T(G_{v; u_1v_1})$. The converse is trivial.

Theorems 2 and 3 answer the main question mentioned at the end
of the introduction, namely, when is $H$ total. The answer is that we try appropriate $v, u_1, v_1$, find $G_{v, u_1 v_1}$ by the algorithm provided in the proof of Theorem 2, and then see if $T(G_{v, u_1 v_1})$ is $H$.

The characterization of disconnected total graphs is reduced to that of connected ones since if $G$ has $m$ components, then $T(G)$ consists of $m$ components each of which is the total graph of a component of $G$. The converse is also true. Thus to completely characterize total graphs it remains to give a characterization of $T(C)$ and $T(K)$. The following results are obtained in [7] and for completeness we state them next.

(i) Let $G$ be a connected regular graph of degree 4. The only such graph with fewer than seven vertices is $T(K_3)$ which is the total graph of each of its triangles. If $|V(G)| \geq 7$, then $G$ is total if and only if: (a) $|V(G)| = 2n$, $n$ a positive integer, (b) $V(G)$ is the disjoint union of two sets each inducing a cycle $C$ of order $n$, and (c) $G$ is the total graph of $C$.

(ii) Let $v$ be a vertex of a graph $H$, and let $G$ be a maximal complete graph contained in the subgraph induced by $N(v)$. Then $H = T(K)$ if and only if $H = T(G)$.

The procedure given in the proof of Theorem 2 for determining whether or not a connected graph $H$, $H \neq T(C)$, $T(K)$, is total seems rather long. There are often easy ways to fix $v$ or to eliminate many possibilities for $u_1$ and $V_1$. For example, if $H$ has odd vertices, then $v$ might be taken to be an odd vertex of $H$.

The following theorem is useful for the determination of those graphs which are total graphs of connected graphs having vertices of degree 1.

**Theorem 4.** Let $H$ be a connected nonregular total graph with a vertex $v_1$ of degree 2 adjacent with vertices $v$ and $u_1$ of $H$. Then:

(a) $\deg u_1 \neq \deg v$, say $\deg v < \deg u_1$;

(b) $v$ is nonspecial while $u_1$ and $v_1$ are the special vertices of $H$ having the property that $v$ corresponds, under the total graph function, to the edge $u_1 v_1$.

**Proof.** It is clear that the degree of each nonspecial vertex of $H$ is greater than two. Thus $v_1$ is special. Hence either $u_1$ or $v$ is nonspecial. Assume $\deg u_1 = \deg v$. Then $\deg u_1 = \frac{1}{2} \deg v + 1$, or $\deg v = \frac{1}{2} \deg u_1 + 1$, both implying that $H$ is regular. Thus we may assume that $\deg v < \deg u_1$.

To prove part (b) suppose that $u_1$ is nonspecial. Then $\deg u_1 = \frac{1}{2} \deg v + 1$ and the inequality $\deg v < \deg u_1$ implies the impossible in-
equality $\deg v_1 \leq 1$. Hence $v$ is nonspecial and clearly $v$ corresponds to the edge $u_1v_1$.

Theorems 3 and 4 provide an easy and practical characterization of total graphs of trees.

**References**


Pahlavi University, Shiraz, Iran

Institute for Mathematical Research, The Imperial Scientific Research Organization, Tehran, Iran