A SHORT PROOF OF A THEOREM OF JENNINGS

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ABSTRACT. We give a short proof of the theorem of Jennings that the augmentation ideal of the group ring of a finitely generated torsion free nilpotent group is residually nilpotent.

Jennings [1] proved the residual nilpotence of the augmentation ideal $A_G = \mathrm{Ker}(R[G] \to R)$ for G a finitely generated torsion free nilpotent group and R a field of characteristic zero. This result is of fundamental importance in applying Lie theoretic methods to problems of abstract group theory.

Jennings' proof is computational and utilizes the fact that G has a basis. The following short proof gives the same result for R any (associative) ring, commutative or not. It is based on a lemma of Swan [2].

We need two lemmas, for which we fix the following data:

$$1 \to H \to G \xrightarrow{P} \boldsymbol{Z} \to 1 \quad \text{is exact.}$$

G is a nilpotent group, and R is a ring. We let t be a fixed generator of $\boldsymbol{Z}$. For a given $x \in G$ which maps onto the generator $t \in \boldsymbol{Z}$ we make $R[H]$ a left $R[G]$-module by defining the action of x on H by $x \cdot h = h^x$, and the action of H on H to be left multiplication. This action of course depends on the x chosen.

LEMMA 1 (SWAN [2]). *For each integer m there is an integer k such that $A_G^k \cdot R[H] \subseteq A_H^m$.*

Swan only needed $R = \boldsymbol{Z}$, but his proof is valid for any ring R. One can take $k = m^c$, where c is the class of G.

LEMMA 2. *Suppose $\alpha \in R[G]$ and $\alpha \neq 0$, where G is torsion free and $H \neq 1$. Then there exists an $x \in G$ mapping onto $t \in \boldsymbol{Z}$ such that $\alpha \cdot R[H] \neq 0$ where the module action $R[G] \cdot R[H] \to R[H]$ is defined relative to x.*

PROOF. We may assume without loss of generality that the coefficient of $1 \in G$ in $\alpha = \sum r_i g_i$ is nonzero.

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Let $1, g_1, \cdots, g_r$ be the distinct group elements occurring in α. Note that if $x, y \in G$ map onto $t \in \boldsymbol{Z}$ and $x \neq y$, then $gp\{x\} \cap gp\{y\} = 1$. For if $x^i = y^j$, then $t^i = P(x^i) = P(y^j) = t^j$, so $i = j$; but in a torsion free nilpotent group $x = y$ whenever $x^i = y^i$ for $i \neq 0$. Thus each of $g_1, \cdots, g_r$ lies in at most one such $gp\{x\}$. Since H is infinite there are infinitely many $x \in G$ mapping onto $t \in \boldsymbol{Z}$ so we can choose one with $g_1, \cdots, g_r \notin gp\{x\}$. $\therefore$ $g_1 = h_1 x^{i_1}, \cdots, g_r = h_r x^{i_r}$ where $h_1, \cdots, h_r \in H, h_1, \cdots, h_r \neq 1$. Consider $1 \in R[H]: 1 \cdot 1 = 1$,

$$g_s \cdot 1 = h_s x^{i_s} \cdot 1 = h_s \neq 1, \quad \text{for } s = 1 \cdots r.$$

$\therefore \alpha \cdot 1 \neq 0$ as required.

THEOREM 1 (JENNINGS [1]). *Let G be a finitely generated torsion free nilpotent group, and let R be any ring. Then $\bigcap A_G^n = 0$.*

PROOF BY INDUCTION ON THE TORSION FREE RANK r OF G.

$r = 1$. Then $R[G] = R[\boldsymbol{Z}] \cong R[t^{\pm 1}]$ where t is commuting indeterminate and the result is clear.

$r > 1$. Let $1 \to H \to G \to \boldsymbol{Z} \to 1$ be exact. Rank $H = r - 1$, so $\bigcap A_H^n = 0$ by the inductive hypothesis. It follows that $\bigcap A_G^n = 0$ by applying Lemmas 1 and 2.

REMARK. A $\mathfrak{D}$-group is a group in which each element has a unique nth root for all n. If G is a nilpotent $\mathfrak{D}$-group, then $\gamma_i G / \gamma_{i+1} G$ is a $\boldsymbol{Q}$-vector space for each i and we can define the $\boldsymbol{Q}$-rank of G to be $\sum [\gamma_i G / \gamma_{i+1} G : \boldsymbol{Q}]$. We can mimic the above procedure to prove the analogue of Theorem 1 for nilpotent $\mathfrak{D}$-groups of finite $\boldsymbol{Q}$-rank.

Consider an exact sequence of nilpotent $\mathfrak{D}$-groups

$$1 \to H \to G \to \boldsymbol{Q} \to 1.$$

Since G is a $\mathfrak{D}$-group, for any $x \in G$ and any rational $\beta = m/n$, the fractional power x^β is uniquely defined, and if $x \neq 1$, $G = \bigcup \{Hx^\beta, \beta \in \boldsymbol{Q}\}$.

We fix a nontrivial $t \in \boldsymbol{Q}$ and for a given $x \in G$ which maps onto t we make $R[H]$ a left $R[G]$-module by defining the action of x^β on H by $x^\beta \cdot h = h^{x^\beta}$ and the action of H on H to be left multiplication. As previously, this action depends on the choice of x.

Lemmas 1 and 2 hold in this new context; the proof of Swan's lemma requires no changes and the proof of Lemma 2 is valid as written, provided $gp\{x\}$ and $gp\{y\}$ are replaced by $gp\{x^\beta : \beta \in \boldsymbol{Q}\}$ and $gp\{y^\beta : \beta \in \boldsymbol{Q}\}$ respectively.

Before proving the analogue of Theorem 1 we need a lemma to begin the induction.

LEMMA 3. *F a field of characteristic* 0, $A_Q = (\text{Ker } F[Q] \to F)$. *Then* $\bigcap A_Q^n = 0$.

PROOF. We will express $F[Q]$ as a polynomial ring in the variable t with fractional exponents, i.e., $F[Q] = F[t^\beta : \beta \in Q]$. There is no loss of generality in assuming that F is algebraically closed. Suppose $\sigma = \sum \sigma_i t^{\beta_i}$, $\sigma \neq 0$, $\sigma \in F[Q]$.

By multiplying σ by an appropriate power of t (a unit in $F[Q]$) and letting n be a common denominator for the β_i we can re-express σ as:

$$\sigma = \sigma_m r^m + \sigma_{m-1} r^{m-1} + \cdots + \sigma_0 = \sigma_m (r - \rho_1) \cdots (r - \rho_m)$$

where $r = t^{1/n}$ and the σ_i, ρ_i are in F.

If $k = \#$ of ρ_i which are equal to 1, it is easy to see that $\sigma \notin A_Q^{k+1}$, essentially because 1 is precisely a 1-fold root of any polynomial $r^l - 1$. $\therefore \bigcap A_Q^n = 0$.

Finally, we note that if G is a nilpotent $\mathfrak{D}$-group of Q-rank r, then there is an exact sequence

$$1 \to H \to G \to Q \to 1$$

where H is a nilpotent $\mathfrak{D}$-group of Q-rank $r-1$. Hence, by induction on the Q-rank of G, we have:

THEOREM 2.[1] *Let G be a nilpotent* $\mathfrak{D}$*-group of finite* Q*-rank, and let R be a field of characteristic* 0. *Then* $\bigcap A_G^n = 0$.

The restriction to characteristic 0 is necessary since, for example, in $\mathbf{Z}_p[G]$ the equation $(g^{1/p} - 1)^p = g - 1$ shows that $\bigcap A_G^n = A_G$.

REFERENCES

1. S. A. Jennings, *The group ring of a class of infinite nilpotent groups*, Canad. J. Math. **7** (1955), 169–187. MR **16**, 899.

2. R. Swan, *Representations of polycyclic groups*, Proc. Amer. Math. Soc. **18** (1967), 573–574. MR **35** #4306.

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[1] In B. Hartley (*The residual nilpotence of wreath products*, Mathematics Institute, University of Warwick) it is proved, by methods similar to Jennings', that $A_G = \text{Ker}(\mathbf{Z}[G] \to \mathbf{Z})$ is residually nilpotent if G is any torsion free nilpotent group.