A ROOT OF UNITY OCCURRING IN
PARTITION THEORY

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Abstract. In this paper a new representation is found for the
root of unity occurring in the well-known transformation equation
of the generating function for \( p(n) \), the number of partitions of the
positive integer \( n \).

Let \( p(n) \) denote the number of partitions of the positive integer \( n \). In the transformation equation for the generating function of \( p(n) \)
(see [1] and [3]) there occurs a certain root of unity which we shall
denote by \( \omega(h, k) \). Here \( k \) is a positive integer and \( h \) is an integer
coprime to \( k \). \( \omega(h, k) \) also appears in the exponential sums \( A_k(n) \)
which occur in the infinite series representation of \( p(n) \) due to Rademacher [3]. It should be mentioned, however, that a formula for
\( A_k(n) \) has been found by Selberg which does not depend on \( \omega(h, k) \)
(see [5] and [6]).

In [1] it was shown by Hardy and Ramanujan that
\[
\begin{align*}
\omega(h, k) &= (-h \mid k) \exp\left\{-\pi i \left(\frac{k - 1}{4} + \frac{(k^2 - 1)(2h + H - h^2H)}{12k}\right)\right\} \\
&\quad \text{if } k \text{ is odd, and} \\
\omega(h, k) &= (-k \mid h) \exp\left\{-\pi i \left(\frac{2 - hk - h}{4} + \frac{(k^2 - 1)(2h + H - h^2H)}{12k}\right)\right\} \\
&\quad \text{if } k \text{ is even. Here, and in the sequel, } (a \mid b) \text{ is the Jacobi symbol while} \\
&\text{ } H \text{ is any solution of the congruence } hH \equiv 1 \pmod{k}. \text{ In [2] Rademacher showed that } \omega(h, k) = \exp\{\pi is(h, k)\} \text{ where } s(h, k) \text{ is a} \\
&\text{Dedekind sum defined by } s(h, k) = \sum_{u=1}^{k-1} \frac{(u/k)((hu/k))}{((x))} = 0 \text{ if } x \text{ is an integer and } ((x)) = x - \lfloor x \rfloor - 1/2 \text{ otherwise.}
\end{align*}
\]

The purpose of the present note is to present still another repre-
sentation of \( \omega(h, k) \) which appears to be somewhat simpler to handle
in computations than those just stated. Thus, we shall prove the following

**Theorem.** If \( k \) is odd then

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\(\omega(h, k) = (h \mid k)i^{(k-1)/2} \exp\{2\pi i(q - h')/gk\}\); 

if \(k\) is even then 

\[\omega(h, k) = (k \mid h)i^{(k+1)/2} \exp\{2\pi i(q - h')/gk\}.\]

If \(J = (k, 3)\) then \(g = J\) or \(8J\) according as \(k\) is odd or even. \(h'\) is any solution of \(hh' = -1\) (mod \(gk\)), and \(q\) is any solution of \(24q/g = 1\) (mod \(gk\)). In (2) \(h' \equiv b\) (mod \(8\)), and the branch of \(i^{2(k+1)/2}\) is that corresponding to the principal value of \(\log z\).

Our proof is based on four lemmas concerning the Dedekind sums \(s(h, k)\). These are essentially Theorems 17, 18, 19 in [4].

**Lemma 1.** \(12ks(h, k) \equiv 0 \pmod{3}\) if and only if \(J = 1\).

**Lemma 2.** \(12ks(h, k) \equiv -h' \pmod{Jk}\).

**Lemma 3.** If \(k\) is odd, then \(12ks(h, k) \equiv k + 1 - 2(h \mid k) \pmod{8}\).

**Lemma 4.** If \(k = 2^aK, a > 0\) and \(K\) odd, then 

\[12ks(h, k) \equiv h - h' - h'^2 - 3h'k + 6h'k(h \mid k) \pmod{2^{a+4}}.\]

Now suppose first that \(k\) is odd and let \(f = 24/J\). Then

\[12ks(h, k) \equiv 9k + 9 + 6(h \mid k) \pmod{f}.\]

For if \(f = 8\) then (3) follows immediately from Lemma 3, while if \(f = 24\) then (3) follows from Lemmas 1, 3 and the Chinese Remainder Theorem.

If we define the integers \(q\) and \(r\) by the congruences

\[fq \equiv 1 \pmod{Jk}, \quad kr \equiv 1 \pmod{f},\]

then it follows from Lemma 2, (3), (4) that 

\[12ks(h, k) \equiv kr(9k + 9 + 6(h \mid k)) +fq(h - h') \pmod{24k}.\]

Therefore,

\[\omega(h, k) = \exp\{2\pi i(12ks(h, k)/24k)\}\]

\[= \exp\{2\pi i(r(9k + 9 + 6(h \mid k))/24 + q(h - h')/Jk)\}.\]

Since \(9 = -9 + 18\) and since \(2r(18 + 6(h \mid k))/24\) is even or odd according as \((h \mid k) = 1\) or \((h \mid k) = -1\), respectively, we see that 

\[\omega(h, k) = (h \mid k) \exp\{2\pi i(h - h')/Jk\} \exp\{3\pi ir(k - 1)/4\}.\]

From (4) we have \(r \equiv k \pmod{8}\), and since \(\exp\{3\pi i(k - 1)/4\} = i^{(k-1)/2}\) the proof of (1) is complete.

If \(k\) is even let \(f = 24/8J\). Then
is immediate if \( f = 1 \) and follows from Lemma 1 if \( f = 3 \). From Lemmas 2 and 4 we have

\[
12ks(h, k) \equiv h - h'(1 + k^2 + 3k - 6(k \mid h)) \pmod{8Jk},
\]

and if \( q \) is defined by the congruence

\[
fq \equiv 1 \pmod{8Jk}
\]

we see from (5) and (6) that

\[
12ks(h, k) \equiv fq(h - h'(1 + k^2 + 3k - 6(k \mid h))) \pmod{24k}.
\]

Therefore,

\[
\omega(h, k) = \exp\left\{2\pi i(12ks(h, k)/24k)\right\} = \exp\left\{2\pi iq(h - h'(1 + k^2 + 3k - 6(k \mid h))/8Jk\right\}.
\]

Since \( 3k \equiv -3k - 18k \pmod{8Jk} \), and since \( 2qh'(18k + 6k(k \mid h))/8Jk \) is even or odd according as \((k \mid h) = 1\) or \((k \mid h) = -1\), we see that

\[
\omega(h, k) = (k \mid h) \exp\left\{2\pi i qh'(3 - k)/8J\right\} \exp\left\{2\pi i q(h - h')/8Jk\right\}.
\]

If \( J = 1 \) then \( f = 3 \), and from (7) we have \( q \equiv 3 \pmod{8} \) so that \( qh'(3 - k) \equiv h'(9 - 3k) \equiv h'(1 + k) \pmod{8} \). If \( J = 3 \) then \( f = 1 \), and from (7) we have \( q \equiv 9 \pmod{8} \) so that \( qh'(1 - k/3) \equiv h'(9 - 3k) \equiv h'(1 + k) \pmod{8} \). Since \( \exp\left\{2\pi i h'(1 + k)/8\right\} = \exp\left\{\pi ib(1 + k)/4\right\} \) if \( h' \equiv b \pmod{8} \) we see that (2) follows from (8) and the proof of the theorem is complete.

We remark in closing that although \( \omega(h, k) \) is almost always referred to in the literature as a 24kth root of unity it is obvious from our theorem that it is also a 12kth root of unity.

ADDED IN PROOF. Since this paper was accepted for publication it has been brought to the author’s attention that Professor T. M. Apostol observed that \( \omega(h, k) \) is a 12kth root of unity in his doctoral dissertation (University of California at Berkeley, 1948). His observation, however, was not published.

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