ON THE IDEAL STRUCTURE OF THE ALGEBRA OF RADIAL FUNCTIONS

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ABSTRACT. Let L denote the convolution Banach algebra of integrable functions defined on $\boldsymbol{R}^n$ and let L_r consist of the subalgebra of radial functions. If I is a closed ideal of L, the zero-set of I is defined by $Z(I)=\{y|\hat{f}(y)=0$ for all $f\in I\}$ where $\hat{f}$ is the Fourier transform of f. The following theorem is proved. If I_1 and I_2 are closed ideals of L_r such that $I_1\subset I_2$ ($\subset$ denotes proper inclusion) then there is a closed ideal I such that $I_1\subset I\subset I_2$.

Let n be a fixed positive integer, and let L denote the Banach algebra of integrable functions defined on $\boldsymbol{R}^n$ with the usual norm and convolution. (The practice of identifying two functions which agree almost everywhere will be followed.) A function f defined on $\boldsymbol{R}^n$ is said to be *radial* if $f(\mathbf{x})=\phi(|\mathbf{x}|)$ for some function ϕ defined on $[0, \infty)$ and for almost every $\mathbf{x}$ in $\boldsymbol{R}^n$; L_r will denote the space of radial functions contained in L. A function in L is radial if and only if its Fourier transform is a radial function (see [1, pp. 69–79]), so L_r is a Banach algebra. If I is a closed ideal of L or of L_r, let

$$Z(I) = \{y \mid \hat{f}(y) = 0 \text{ for every } f \in I\}.$$

$Z(I)$ is called the zero-set of I.

Helson showed in [2] that if I_1 and I_2 are closed ideals of L such that $Z(I_1)=Z(I_2)$ and $I_1\subset I_2$ ($\subset$ denotes proper inclusion), then there is a closed ideal I such that $I_1\subset I\subset I_2$. In the present paper Helson's theorem will be used to prove the following:

THEOREM. *If I_1 and I_2 are closed ideals of L_r such that $Z(I_1)=Z(I_2)$ and $I_1\subset I_2$, then there is a closed ideal I of L_r such that $I_1\subset I\subset I_2$.*

The proof of the theorem will be given later; it is necessary, first, to examine how L_r sits in L.

Let $d\mu$ be the positive measure of unit mass distributed uniformly on the hypersphere $S=\{\mathbf{x}|\mathbf{x}\in\boldsymbol{R}^n$ and $|\mathbf{x}|=1\}$, and set

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$$f_r(\mathrm{x}) = \int f(|\mathrm{x}|y)d\mu(y). \tag{1}$$

The integral must exist for almost every x by Fubini's theorem since $\boldsymbol{R}^n$ can be thought of as a product of two measure spaces: one being $\boldsymbol{S}$ with the measure $d\mu$ and the other being $[0, \infty)$ with the measure $c\rho^{n-1}d\rho$ where $d\rho$ is Lebesgue measure and c is the surface area of $\boldsymbol{S}$. It also follows from Fubini's theorem that f_r is in L. Define

$$L_0 = \{f \mid f \in L \text{ and } f_r(\mathrm{x}) = 0 \text{ for almost every } \mathrm{x} \text{ in } \boldsymbol{R}^n\};$$

finally let $f_0(\mathrm{x}) = f(\mathrm{x}) - f_r(\mathrm{x})$. The following lemmas list some properties of L_r, L_0, f_r, and f_0.

LEMMA 1. *The map $f \to f_r$ is a continuous projection with unit norm, hence its null space L_0 is closed and so $L = L_0 \oplus L_r$.*

The proof of Lemma 1 follows from the easily verified facts that $\|f_r\| \leqq \|f\|$ and that $f = f_r$ if f is radial.

A thorough discussion of this decomposition can be found in [3].

LEMMA 2. *f is contained in L_0 if and only if*

$$\int \hat{f}(\rho y)d\mu(y) = 0 \qquad (\rho > 0). \tag{2}$$

PROOF. Application of Fubini's theorem yields

$$\int \hat{f}(\rho y)d\mu(y) = \int_{R^n} f(\mathrm{x})d\mathrm{x}\left\{\int \exp(i\mathrm{x}\cdot\rho y)d\mu(y)\right\}$$

$$= \int_{R^n} f(\mathrm{x})K(\mathrm{x})dx,$$

where $K(\mathrm{x})$ is the value of the inner integral. $K(\mathrm{x})$ is a radial function because μ is a weak limit of radial functions and $K(\mathrm{x})$ is the Fourier-Stieltjes transform of μ, or see [1, pp. 69–79]. Conversion of the last integral into hyperspherical coordinates yields (2).

To prove the converse, suppose (2) holds for some f in L. Then

$$\int \hat{f}_r(|\mathrm{x}|y)d\mu(y) + \int \hat{f}_0(|\mathrm{x}|y)d\mu(y) = 0.$$

The second integral vanishes by the first part of this lemma since f_0 is in L_0, and the value of the first integral is $\hat{f}_r(\mathrm{x})$. Thus $\hat{f}_r = 0$, so $f_r = 0$ and hence f is in L_0.

LEMMA 3. *The convolution of a function in L_0 and a function in L_r is contained in L_0.*

PROOF. Suppose f is in L_r and g is in L_0. Then for each $\mathbf{x}$ in $\boldsymbol{R}^n$

$$\int (f * g)^\wedge(|\mathbf{x}|y)d\mu(y) = \int \hat{f}(|\mathbf{x}|y)\hat{g}(|\mathbf{x}|y)d\mu(y)$$

$$= \hat{f}(\mathbf{x}) \int \hat{g}(|\mathbf{x}|y)d\mu(y)$$

$$= 0$$

by Lemma 2.

LEMMA 4. *Let I be a closed ideal of L_r and let K be the closed ideal of L generated by I. Then $I = K \cap L_r$ and $Z(I) = Z(K)$.*

PROOF. I is contained in K, hence in $K \cap L_r$. The fact that $K \cap L_r$ is contained in I will follow from the stronger fact that if f is in K, then f_r is in I. Suppose

$$(3) \quad f = h + \sum_{i=1}^{m} h_i * g_i \qquad (h \in I,\ h_i \in I,\ g_i \in L;\ i = 1, 2, \cdots, m).$$

Then $f = h + \sum_{i=1}^{m} h_i * (g_i)_r + \sum_{i=1}^{m} h_i * (g_i)_0$. The second sum is contained in L_0 by Lemma 3 and the first sum is contained in I because I is an ideal of L_r. Finally, the first sum plus h is f_r by Lemma 1; hence if f has the form (3) then f_r is in I. If f is any function in K, there is a sequence $\{f_k\}$ of finite linear combinations of the form of (3) such that f_k converges to f in L. The transformation of f into f_r is continuous on L so $\{(f_k)_r\}$ converges to f_r. Since I is closed, it must contain f_r.

Finally $Z(K) \subseteq Z(I)$ because $I \subseteq K$ and $Z(K) = Z(I)$ since finite linear combinations of the form of (3) are dense in K.

PROOF OF THEOREM. Let K_1 and K_2 be the closed ideals of L generated by I_1 and I_2 respectively. Then $K_1 \subset K_2$ by Lemma 4 because $K_1 \cap L_r = I_1 \subset I_2 = K_2 \cap L_r$, and

$$Z(K_1) = Z(I_1) = Z(I_2) = Z(K_2).$$

By Helson's theorem there must be a closed ideal K such that $K_1 \subset K \subset K_2$. Since K_2 is the ideal generated by I_2, it follows that $K \cap L_r \subset I_2$. The inclusion $I_1 \subset K \cap L_r$ is not immediate. Suppose there is no closed ideal K of L such that

$$(4) \qquad I_1 \subset K \cap L_r \subset I_2,$$

then define $\mathcal{K}$ to be the collection of all closed ideals of L such that

$$K_1 \subset K \subset K_2 \quad \text{and} \quad K \cap L_r = I_1.$$

Let $\mathcal{K}$ be ordered by inclusion and let K^* be the union of all the ideals in a maximal chain of $\mathcal{K}$.

K^* is contained in $\mathcal{K}$. To see this let J be the closure in L of K^*. If f is in $K^* \cap L_r$, then f is in $K \cap L_r$ for some K in $\mathcal{K}$ so f is in I_1; thus $K^* \cap L_r = I_1$ so $J \cap L_r = I_1$. Since $J \cap L_r = I_1$, it follows that $J \subset K_2$. Since K^* is a union of elements of $\mathcal{K}$ it follows that $K_1 \subset J \subset K_2$. Thus J is in $\mathcal{K}$ and so $K^* = J$ by the construction of K^*; hence, K^* is in $\mathcal{K}$. It also follows that $Z(K^*) = Z(K_2)$ because K^* lies between K_1 and K_2.

Helson's theorem can now be invoked to guarantee the existence of an ideal K^{**} such that $K^* \subset K^{**} \subset K_2$. Since $K^{**} \subset K_2$ it follows that $K^{**} \cap L_r \subset I_2$, and the proper inclusion $I_1 \subset K^{**} \cap L_r$ holds by the construction of K^*. Thus K^{**} contradicts our assumption that no ideal of L satisfies (4).

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