ON THE IDEAL STRUCTURE OF THE ALGEBRA OF RADIAL FUNCTIONS

ALAN SCHWARTZ

Abstract. Let $L$ denote the convolution Banach algebra of integrable functions defined on $\mathbb{R}^n$ and let $L_r$ consist of the sub-algebra of radial functions. If $I$ is a closed ideal of $L$, the zero-set of $I$ is defined by $Z(I) = \{ y \mid \hat{f}(y) = 0 \text{ for all } f \in I \}$ where $\hat{f}$ is the Fourier transform of $f$. The following theorem is proved. If $I_1$ and $I_2$ are closed ideals of $L_r$ such that $I_1 \subseteq I_2$ (denotes proper inclusion) then there is a closed ideal $I$ such that $I_1 \subseteq I \subseteq I_2$.

Let $n$ be a fixed positive integer, and let $L$ denote the Banach algebra of integrable functions defined on $\mathbb{R}^n$ with the usual norm and convolution. The practice of identifying two functions which agree almost everywhere will be followed. A function $f$ defined on $\mathbb{R}^n$ is said to be radial if $f(x) = \phi(|x|)$ for some function $\phi$ defined on $[0, \infty)$ and for almost every $x$ in $\mathbb{R}^n$; $L_r$ will denote the space of radial functions contained in $L$. A function in $L$ is radial if and only if its Fourier transform is a radial function (see [1, pp. 69–79]), so $L_r$ is a Banach algebra. If $I$ is a closed ideal of $L$ or of $L_r$, let

$Z(I) = \{ y \mid \hat{f}(y) = 0 \text{ for every } f \in I \}$.

$Z(I)$ is called the zero-set of $I$.

Helson showed in [2] that if $I_1$ and $I_2$ are closed ideals of $L$ such that $Z(I_1) = Z(I_2)$ and $I_1 \subseteq I_2$ (denotes proper inclusion), then there is a closed ideal $I$ such that $I_1 \subseteq I \subseteq I_2$. In the present paper Helson's theorem will be used to prove the following:

Theorem. If $I_1$ and $I_2$ are closed ideals of $L_r$ such that $Z(I_1) = Z(I_2)$ and $I_1 \subseteq I_2$, then there is a closed ideal $I$ of $L_r$ such that $I_1 \subseteq I \subseteq I_2$.

The proof of the theorem will be given later; it is necessary, first, to examine how $L_r$ sits in $L$.

Let $d\mu$ be the positive measure of unit mass distributed uniformly on the hypersphere $S = \{ x \mid x \in \mathbb{R}^n \text{ and } |x| = 1 \}$, and set

Received by the editors September 5, 1969.

AMS 1969 subject classifications. Primary 4240, 4258.

Key words and phrases. Convolution algebra, Fourier transform, ideal structure, radial functions, zero-sets.

1 Supported by an Assistant Professor Research Grant at the University of Missouri, St. Louis.
\[ f_r(x) = \int f(\frac{1}{|x|}y) d\mu(y). \]

The integral must exist for almost every \( x \) by Fubini's theorem since \( \mathbb{R}^n \) can be thought of as a product of two measure spaces: one being \( S \) with the measure \( d\mu \) and the other being \([0, \infty)\) with the measure \( c \mu^{n-1} dp \) where \( dp \) is Lebesgue measure and \( c \) is the surface area of \( S \). It follows from Fubini's theorem that \( f_r \) is in \( L \). Define

\[ L_0 = \{ f \mid f \in L \text{ and } f_r(x) = 0 \text{ for almost every } x \in \mathbb{R}^n \}; \]

finally let \( f_0(x) = f(x) - f_r(x) \). The following lemmas list some properties of \( L_r, L_0, f_r, \) and \( f_0 \).

**Lemma 1.** The map \( f \rightarrow f_r \) is a continuous projection with unit norm, hence its null space \( L_0 \) is closed and so \( L = L_0 \oplus L_r \).

The proof of Lemma 1 follows from the easily verified facts that \( \|f_r\| \leq \|f\| \) and that \( f = f_r \) if \( f \) is radial.

A thorough discussion of this decomposition can be found in [3].

**Lemma 2.** \( f \) is contained in \( L_0 \) if and only if

\[ \int f(\rho y) d\mu(y) = 0 \quad (\rho > 0). \]

**Proof.** Application of Fubini's theorem yields

\[
\int f(\rho y) d\mu(y) = \int_{\mathbb{R}^n} f(x) dx \left\{ \int \exp(i x \cdot \rho y) d\mu(y) \right\} \\
= \int_{\mathbb{R}^n} f(x) K(x) dx,
\]

where \( K(x) \) is the value of the inner integral. \( K(x) \) is a radial function because \( \mu \) is a weak limit of radial functions and \( K(x) \) is the Fourier-Stieltjes transform of \( \mu \), or see [1, pp. 69–79]. Conversion of the last integral into hyperspherical coordinates yields (2).

To prove the converse, suppose (2) holds for some \( f \) in \( L \). Then

\[
\int f_r^r(\frac{1}{|x|} y) d\mu(y) + \int f_0^r(\frac{1}{|x|} y) d\mu(y) = 0.
\]

The second integral vanishes by the first part of this lemma since \( f_0 \) is in \( L_0 \), and the value of the first integral is \( f_r(x) \). Thus \( f_r = 0 \), so \( f_r = 0 \) and hence \( f \) is in \( L_0 \).
**Lemma 3.** The convolution of a function in $L_0$ and a function in $L_r$ is contained in $L_0$.

**Proof.** Suppose $f$ is in $L_r$ and $g$ is in $L_0$. Then for each $x$ in $\mathbb{R}^n$

$$\int (f \ast g) \, \mu(y) = \int f(x) \, g(x) \, \mu(y)$$

by Lemma 2.

**Lemma 4.** Let $I$ be a closed ideal of $L_r$ and let $K$ be the closed ideal of $L$ generated by $I$. Then $I = K \cap L_r$ and $Z(I) = Z(K)$.

**Proof.** $I$ is contained in $K$, hence in $K \cap L_r$. The fact that $K \cap L_r$ is contained in $I$ will follow from the stronger fact that if $f$ is in $K$, then $f_r$ is in $I$. Suppose

$$f = h + \sum_{i=1}^m h_i \ast g_i \quad (h \in I, h_i \subseteq I, g_i \in L_0; i = 1, 2, \ldots, m).$$

Then $f = h + \sum_{i=1}^m h_i \ast (g_i)_r + \sum_{i=1}^m h_i \ast (g_i)_0$. The second sum is contained in $L_0$ by Lemma 3 and the first sum is contained in $I$ because $I$ is an ideal of $L_r$. Finally, the first sum plus $h$ is $f_r$ by Lemma 1; hence if $f$ has the form (3) then $f_r$ is in $I$. If $f$ is any function in $K$, there is a sequence $\{f_k\}$ of finite linear combinations of the form of (3) such that $f_k$ converges to $f$ in $L$. The transformation of $f$ into $f_r$ is continuous on $L$ so $\{(f_k)_r\}$ converges to $f_r$. Since $I$ is closed, it must contain $f_r$.

Finally $Z(K) \subseteq Z(I)$ because $I \subseteq K$ and $Z(K) = Z(I)$ since finite linear combinations of the form of (3) are dense in $K$.

**Proof of Theorem.** Let $K_1$ and $K_2$ be the closed ideals of $L$ generated by $I_1$ and $I_2$ respectively. Then $K_1 \subseteq K_2$ by Lemma 4 because $K_1 \cap L_r = I_1 \subseteq I_2 = K_2 \cap L_r$, and

$$Z(K_1) = Z(I_1) = Z(I_2) = Z(K_2).$$

By Helson's theorem there must be a closed ideal $K$ such that $K_1 \subseteq K \subseteq K_2$. Since $K_2$ is the ideal generated by $I_2$, it follows that $K \cap L_r \subseteq I_2$. The inclusion $I_1 \subseteq K \cap L_r$ is not immediate. Suppose there is no closed ideal $K$ of $L$ such that

$$I_1 \subseteq K \cap L_r \subseteq I_2,$$

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then define \( \mathcal{K} \) to be the collection of all closed ideals of \( L \) such that
\[
K_1 \subset K \subset K_2 \quad \text{and} \quad K \cap L_r = I_1. 
\]

Let \( \mathcal{K} \) be ordered by inclusion and let \( K^* \) be the union of all the ideals in a maximal chain of \( \mathcal{K} \).

\( K^* \) is contained in \( \mathcal{K} \). To see this let \( J \) be the closure in \( L \) of \( K^* \).
If \( f \) is in \( K^* \cap L_r \), then \( f \) is in \( K \cap L_r \) for some \( K \) in \( \mathcal{K} \) so \( f \) is in \( I_1 \); thus \( K^* \cap L_r = I_1 \) so \( J \cap L_r = I_1 \). Since \( J \cap L_r = I_1 \), it follows that \( J \subset K_2 \). Since \( K^* \) is a union of elements of \( \mathcal{K} \) it follows that \( K_1 \subset J \subset K_2 \).
Thus \( J \) is in \( \mathcal{K} \) and so \( K^* = J \) by the construction of \( K^* \); hence, \( K^* \) is in \( \mathcal{K} \). It also follows that \( Z(K^*) = Z(K_2) \) because \( K^* \) lies between \( K_1 \) and \( K_2 \).

Helson's theorem can now be invoked to guarantee the existence of an ideal \( K^{**} \) such that \( K^* \subset K^{**} \subset K_2 \). Since \( K^{**} \subset K_2 \) it follows that \( K^{**} \cap L_r \subset I_2 \), and the proper inclusion \( I_1 \subset K^{**} \cap L_r \) holds by the construction of \( K^* \). Thus \( K^{**} \) contradicts our assumption that no ideal of \( L \) satisfies (4).

\textbf{References}