A COMBINATORIAL PROBLEM AND CONGRUENCES
FOR THE RAYLEIGH FUNCTION

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ABSTRACT. Let $z$ be a positive integer and let $m$ be the number
of nonzero terms in the base 2 expansion of $z$. Define $f(z, s)$ as the
number of positive integers $r \leq z/2$ such that the number of non-
zero terms in the base 2 expansion of $r$ plus the number of nonzero
terms in the base 2 expansion of $z-r$ is equal to $m+s$. We find
formulas for $f(z, s)$ and show how these formulas can be used in
proving congruences for the Rayleigh function.

1. Introduction. We shall assume that

$$z = 2^{e_1} + \cdots + 2^{e_m}, \quad e_1 > \cdots > e_m \geq 0.$$ 

For $s \geq 0$ define $f(z, s)$ as the number of positive integers $r \leq z/2$ such
that if

$$r = 2^{b_1} + \cdots + 2^{b_s}, \quad b_1 > \cdots > b_s \geq 0,$$

$$z - r = 2^{c_1} + \cdots + 2^{c_w}, \quad c_1 > \cdots > c_w \geq 0,$$

then $v+w = m+s$.

In §4 we point out that $f(z, s)$ is equal to the number of binomial
coefficients ($\binom{\cdot}{\cdot}$), $0<r \leq z/2$, divisible by $2^s$ but not by $2^{s+1}$. The main
purpose of this paper is to consider the problem of evaluating $f(z, s)$
for $s \geq 0$ and to show how this function can be used for proving con-
gruences for the Rayleigh function, $\sigma_{2n}(v)$. We note that in [2] it was
proved that $f(z, 0) = 2^{m-1}-1$.

The Rayleigh function has been the subject of a number of papers
by Kishore (see [3] and [4] for example). It can be defined by means
of the recurrence formula

$$\sigma_{2n}(v) = \sum_{i=1}^{n-1} \sigma_{2i}(v)\sigma_{2n-2i}(v),$$

where $\sigma_2(v) = 1/4(v+1)$. It is known that

$$\sigma_{2n}(1/2) = (-1)^{n-1}2^{2n-1}B_{2n}/(2n)!,$$

$$\sigma_{2n}(-1/2) = (-1)^{n}2^{2n-2}G_{2n}/(2n)!$$

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where $B_{2n}$ is the $2n$th Bernoulli number and $G_n = 2(1 - 2^n)B_n$.

In this paper we prove congruences (mod 8) and (mod 16) for $2^{
u}a_{2n}(a/b)$, when neither the numerator nor denominator of $2^{
u}a_{2n}(a/b)$ is divisible by 2.

2. A combinatorial problem. The notation of the introduction will be used in this section, and we will make use of the function $\alpha(j)$ defined by means of

\[ \alpha(j) = \begin{cases} 2 & \text{if } s + m = 2j, \\ 1 & \text{if } s + m > 2j. \end{cases} \]

**Theorem 2.1.** If $e_i - e_{i+1} > s > 0$ for $i = 1, \ldots, m-1$ and $e_m \geq s$, then

\[ f(z, s) = \sum_{j=1}^{m} \binom{m}{j} \binom{s - 1}{j - 1} \alpha(j) 2^{s+m-2j-1}. \]

**Proof.** To find $r$ and $z - r$ satisfying (1.1) we first pick $j$ elements from $e_1, \ldots, e_m$. Call them $d_1, \ldots, d_j$. Consider any composition of $s$ into $j$ parts (see [5, p. 107]):

\[ s = s_1 + \cdots + s_j. \]

Then for $i = 1, \ldots, j$ we have

\[ 2^{d_i} = 2^{d_i-s_i} + \sum_{h=0}^{s_i-1} 2^{d_i-s_i+h}. \]

The exponents $b$ and $c$ of (1.1) will be chosen from the $s+j$ numbers $d_i - s_i + h$ and the $m-j$ others which were not picked. There are $2^{m+s-2j}$ ways to divide these numbers between $r$ and $z-r$. There are $\binom{s}{j}$ ways to pick $d_1, \ldots, d_j$ and there are $\binom{s-j}{j-1}$ compositions of $s$ into $j$ parts [5, p. 124]. We divide our final answer by 2 (except when $s+m=2j$) to eliminate $r > z/2$.

**Theorem 2.2.** If $e_i - e_{i+1} > s > 0$ for $q$ terms $e_i$, if $e_i - e_{i+1} = s$ for $m-q-1$ terms and $e_m \geq s$, then

\[ f(z, s) = (q + 1)2^{s+m-3} + \sum_{j=2}^{m} \binom{m}{j} \binom{s - 1}{j - 1} \alpha(j) 2^{s+m-2j-1}. \]

**Theorem 2.3.** If $e_i - e_{i+1} > s > 0$ for $q$ terms $e_i$, if $e_i - e_{i+1} = s$ for $m-q-1$ terms and $e_m = u$, $0 \leq u < s$, then
Proof. We use the method of Theorem 2.1. Pick \( j \) terms from \( e_1, \ldots, e_m \) and call them \( d_1, \ldots, d_j \). The term \( q^{2^s+m-3} \) in the formula corresponds to the case \( j = 1 \). For \( 2 \leq j \leq m \) we proceed exactly as we did in Theorem 2.1. We observe, however, that certain compositions of \( s \) into \( j \) parts will not be allowed when \( 2 \leq j \leq s-u \) and \( e_m \) is picked as one of the \( d \)'s. When this happens, there are \( \binom{s-1}{j-1} \) ways of picking the remaining \( d \)'s, and we need to find the number of compositions of \( s \) into \( j \) parts \( s_1 + \cdots + s_j \) such that \( s_1 > u \). An enumerating generating function for such compositions is

\[
(x + x^2 + \cdots + x^u)^{j-1}.
\]

From this we determine the number of such compositions to be \( \binom{s-1}{j-1} \). The formula of Theorem 2.3 now follows.

3. Congruences for the Rayleigh function. We now show how the formulas of §2 can be used in proving congruences for the Rayleigh function.

Let \( a/b \) be a rational number, \( a \) odd, \( b \) even, \( b = (2k+1)2^t \), and let \( y = 2n + 1 + (1-2n)t - m \), where \( m \) is the number of nonzero terms in the base 2 expansion of \( 2n \). In [2] the following results were proved.

**Lemma 3.1.** If \( 2n = 2e_1 + \cdots + 2e_m, e_1 > \cdots > e_m \geq 1 \), then

\[
2^u\sigma_{2n}(a/b) \equiv 1 \pmod{2}.
\]

**Lemma 3.2.** If \( 2n = 2e_1, e_1 \geq 2 \), then

\[
2^u\sigma_{2n}(a/b) \equiv 5(2k+1)/a \pmod{8}.
\]

**Lemma 3.3.** If \( 2n = 2e_1 + \cdots + 2e_m, e_i - e_{i+1} > 1 \) for \( i = 1, \ldots, m-1 \) and \( e_m \geq 1 \), then

\[
2^u\sigma_{2n}(a/b) \equiv (-1)^{m+1}(2k+1)/a \pmod{4}.
\]

We can now prove the following theorems.

**Theorem 3.1.** If \( 2n = 2e_1 + 2e_2, e_1 - e_2 > 2, e_2 \geq 2 \), then

\[
2^u\sigma_{2n}(a/b) \equiv 7(2k+1)/a \pmod{8}.
\]

**Proof.** From (1.2) and Lemmas 3.1, 3.2 and 3.3, we have
Theorem 3.2. If \( 2n = 2^n + 2^m + 2^s, \ e_i - e_{i+1} > 2 \) for \( i = 1, 2 \) and \( e_s > 2 \), then

\[
2^v \sigma_{2n}(a/b) = 7(2^m + 1)/a \quad (\text{mod } 8).
\]

Proof. We have

\[
2^v \sigma_{2n}(a/b) = [3f(2n, 0) + 2f(2n, 1) + 4f(2n, 2)](2^k + 1)/a
\]

\[
= 7(2^m + 1)/a \quad (\text{mod } 8).
\]

Theorem 3.3. If \( 2n = 2^n + \cdots + 2^m, \ e_i - e_{i+1} > 2 \) for \( i = 1, \cdots, m-1 \) and \( e_m > 2 \), then

\[
2^v \sigma_{2n}(a/b) = 5(2^m + 1)/a \quad (\text{mod } 8).
\]

Proof. The proof is by induction on \( m \). The theorem is true for \( m = 1, 2, 3 \). Assume it is true for \( 1, 2, 3, \cdots, m-1 \). If \( m \) is even we have

\[
2^v \sigma_{2n}(a/b) = [3f(2n, 0) + 2f(2n, 1) + 4f(2n, 2)](2^k + 1)/a \quad (\text{mod } 8).
\]

We know

\[
f(2n, 0) = 2^{m-1} - 1 = 7 \quad (\text{mod } 8),
\]

\[
f(2n, 1) = m2^{m-2} = 0 \quad (\text{mod } 4),
\]

\[
f(2n, 2) = m2^{m-1} + \binom{m}{2}2^{m-3} = 0 \quad (\text{mod } 2).
\]

Therefore

\[
2^v \sigma_{2n}(a/b) = 7(2^m + 1)/a \quad (\text{mod } 8).
\]

If \( m \) is odd we have

\[
2^v \sigma_{2n}(a/b) = [3f(2n, 0) + 2f(2n, 1) + 4f(2n, 2)](2^k + 1)/a
\]

\[
= 5(2^m + 1)/a \quad (\text{mod } 8).
\]

We can use the method of this section to prove congruences (mod 16). The following theorem is first verified for \( m = 1, 2, 3, 4 \) and then proved by induction on \( m \).

Theorem 3.4. If \( 2n = 2^n + \cdots + 2^m, \ e_i - e_{i+1} > 3 \) for \( i = 1, \cdots, m-1 \) and \( e_m > 3 \), then
2^m \sigma_m(a/b) \equiv 5((2k+1)/a)^3 \pmod{16} \text{ if } m \equiv 3 \pmod{4},
\equiv 7((2k+1)/a)^3 \pmod{16} \text{ if } m \equiv 2 \pmod{4},
\equiv 13((2k+1)/a)^3 \pmod{16} \text{ if } m \equiv 1 \pmod{4},
\equiv 15((2k+1)/a)^3 \pmod{16} \text{ if } m \equiv 0 \pmod{4}.

4. Relationship between \( f(z, s) \) and binomial coefficients. We use the notation of the introduction in this section. Consider the binomial coefficient \( \binom{n}{r} \), \( 0 < r \leq n/2 \). It is well known that the exponent of the highest power of 2 dividing \( \binom{n}{r} \) is \( s = v + w - m \). This follows from the fact that

\[ n = 2^{a_1} + \cdots + 2^{a_i}, \quad a_1 > \cdots > a_i \geq 0, \]

then the exponent of the highest power of 2 dividing \( n! \) is \( n - i \) [1, p. 61]. Hence, by the definition of \( f(z, s) \), we see that \( f(z, s) \) is equal to the number of binomial coefficients \( \binom{n}{r} \), \( 0 < r \leq n/2 \), divisible by \( 2^s \) but not by \( 2^{s+1} \). I am grateful to the referee and to L. Carlitz for suggestions concerning this section.

References


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