SQUARE INTEGRABLE SOLUTIONS OF $L_y = f(t, y)$

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Abstract. Let $L$ be an ordinary linear differential operator and $L^+$ its formal adjoint. It is shown that, under suitable conditions on $f$, all solutions of $L_y = f(t, y)$ are in $L^2(0, \infty)$ provided that all those of $L_y = 0$ and $L^+y = 0$ are.

1. Introduction. For $L$ a regular ordinary linear differential or quasi-differential operator of any order we show, under suitable conditions on $f$, that all solutions of

(1) $L_y = f(t, y)$

are in $L^2(0, \infty)$, provided all solutions of

(2) $L_y = 0$

and

(3) $L^+y = 0$

are, where $L^+$ is the formal adjoint of $L$.

The case $L_y = L^+y = (ry')' + qy$ is a recent result of J. S. Bradley [4]. The linear case with $L_y = L^+y = y'' + qy$ and $f(t, y) = h(t)y$ with $h(t)$ bounded is due to Bellman [3]. See also Halvorsen [6].

2. Notation and preliminaries. We consider first the classical operators $L$ and $L^+$ defined by

$L_y = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y,$

$L^+z = ((-1)^{n-1}p_{n-1}z)' + \cdots + ((-1)^2p_{n-2}z)' + (-1)^n p_nz$,

where the $p_i$'s for $i = 0, \ldots, n - 1$ are continuous real valued functions on $[a, \infty)$. This form of the adjoint equation is used to avoid smoothness assumptions on the coefficients. Denote by $S$ and $S^+$ the solution spaces of $L_y = 0$ and $L^+y = 0$, respectively. Let $y_i$ for $i = 1, \ldots, n$ be the solutions of (2) determined by the initial conditions

$y_i^{(j)}(a) = \delta_{i,j+1}$ for $j = 0, \ldots, n - 1$.  

Received by the editors January 16, 1970.

AMS 1969 subject classifications. Primary 3453; Secondary 4760.

Key words and phrases. Square integrable solutions, differential operators, Gronwall inequality, variation of parameters.
Throughout the paper $\delta$ will denote the Kronecker $\delta$. Define $D_0^+ z = z$, $D_i^+ z = (D_{i-1}^+ z)' + (-1)^i \rho_{n-i} z$ for $i = 1, \cdots, n-1$.

Let $z_i$ for $i = 1, \cdots, n$ be the solutions of (3) determined by

$$D_i z_i (a) = (-1)^{n-i} \delta_{i,n-j} \quad \text{for } j = 0, \cdots, n-1.$$

The proof of the main theorem will be based on two lemmas which we now state.

**Lemma 1.** Suppose $f \in C[a, \infty)$. The solution $y$ of $Ly = f$ satisfying $y'(a) = \alpha_j$, for $j = 0, \cdots, n-1$, is given by

$$y(t) = \sum_{i=1}^n \alpha_i y_i(t) + \int_a^t \sum_{i=1}^n y_i(t) z_i(s)f(s)ds.$$

This representation of solutions of nonhomogeneous equations differs from that found in most textbooks. So, although Lemma 1 follows from Theorems 2 and 3 in [11], we sketch a proof here for the sake of completeness.

**Proof.** Let

$$Y' = FY$$

be the usual vector-matrix formulation of equation (2), i.e. $Y$ is the column vector $[y, y', \cdots, y^{n-1}]$ and $F$ is the matrix having all 1's in the super diagonal, $-\rho_0, -\rho_1, \cdots, -\rho_{n-1}$ in the bottom row from left to right and zeros elsewhere. Note that equation (3) is equivalent to the vector-matrix equation

$$Z' = F^+ Z$$

where $Z$ is the column vector $[D_0^+ z, D_1^+ z, \cdots, D_{n-1}^+ z]$ and $F^+ = - J^{-1} F*J$ with $J = \begin{pmatrix} 1, n+1-j \end{pmatrix}$. Denote by $Y_u$ the unique matrix solution of (5) satisfying $Y(u) = I$ and by $Z_u$ the unique matrix solution of (6) satisfying $Z(u) = I$ where $I$ is the $n \times n$ identity matrix.

Let $M(t, u) = Y_u(t) = (M_{ij}(t, u))$ and $N(t, u) = Z_u(t) = (N_{ij}(t, u))$ for $t \geq a$, $u \geq a$. Noting that $M(t, s) = M(t, u)M(u, s)$ one can readily verify that

$$M(t, u) = J^{-1} N^*(u, t)J \quad \text{for } t \geq a, \quad u \geq a.$$

From the variation of constants formula it follows that

$$y(t) = \sum_{i=1}^n M_{1i}(t, a) \alpha_i + \int_a^t M_{1n}(t, u)f(u)du.$$
As a consequence of (8) and the definitions of $M$, $N$, $y$, and $z$, we can obtain that

$$M_{1n}(t, s) = \sum_{i=1}^{n} M_{1i}(t, a) M_{in}(a, s)$$

$$= \sum_{i=1}^{n} M_{1i}(t, a)(-1)^{i+n} N_{1i}(s, a) = \sum_{i=1}^{n} y_{i}(t) z_{i}(s)$$

and (4) follows.

The second lemma is a very special case of a generalization of the well-known Gronwall inequality due to H. E. Gollwitzer [5] (also see Willett [9] and Willett and Wong [10]).

**Lemma 2.** Let $u, \phi, g, h$ be nonnegative continuous functions on $[a, \infty)$ and suppose that

$$u(t) \leq \phi(t) + g(t) \left( \int_{a}^{t} u^{2}(s) h(s) ds \right)^{1/2} \quad \text{for } t \geq a.$$  

Then

$$u(t) \leq \phi(t) + g(t) \left\{ \int_{a}^{t} 2\phi^{2}(s) h(s) \exp \left[ \int_{s}^{t} 2g(x) h(x) dx \right] ds \right\}^{1/2} \quad \text{for } t \geq a.$$  

**3. The main theorem.** Suppose there exist functions $h$ and $k$ such that

(9) \quad \left| f(t, y) \right| \leq k(t) + h(t) y \quad \text{for } t \geq a, \quad -\infty < y < \infty.

**Theorem 1.** Suppose $S \cup S^{+} \subseteq L^{2}(a, \infty)$ and $kz \in L^{1}(a, \infty)$ for all $z \in S^{+}$ and either

(i) $|h|^{1/2} |y| \in L^{2}(a, \infty)$ for all $y \in S \cup S^{+}$ or

(ii) $hz \in L^{2}(a, \infty)$ for all $z \in S^{+}$.

Then all solutions of (1) are in $L^{2}(a, \infty)$.

**Proof.** We prove first that all solutions of (1) are in $L^{2}(a, \infty)$ if (i) holds. Note that (9) implies all solutions are defined on $[a, \infty)$. Let $y$ be a solution of (1).

By Lemma 1

$$y(t) = \sum_{i=1}^{n} \alpha_{i} y_{i}(t) + \int_{a}^{t} \sum_{i=1}^{n} y_{i}(t) z_{i}(s) f(s, y(s)) ds.$$  

Hence
Letting $C_i = f^\alpha \ |z_i k|$ and the Schwarz inequality we obtain
\begin{equation}
|y(t)| \leq \sum_{i=1}^{n} \left[ C_i + \alpha_i \right] |y_i(t)| \nonumber
\end{equation}
\begin{equation}
+ \sum_{i=1}^{n} |y_i(t)| \left( \int_a^t |z_i(s)| \left( |k(s)| + |h(s)y(s)| \right) ds. \right) \nonumber
\end{equation}

Letting $D_i = \left( \int_a^\alpha |z_i|^2 |h| \right)^{1/2}$, then $|y(t)| \leq \phi(t) + g(t) \left( \int_a^\alpha y^2 |h| \right)^{1/2}$ where $\phi(t) = \sum_{i=1}^{n} \left( C_i + \alpha_i \right) |y_i(t)|$ and $g(t) = \sum_{i=1}^{n} D_i |y_i(t)|$. It follows from Lemma 2 that
\begin{equation}
|y(t)| \leq \phi(t) + g(t) \left\{ \int_a^t 2\phi^2(s) |h(s)| \nonumber
\end{equation}
\begin{equation}
\cdot \exp \left[ \int_s^{s'} 2g^2(x) \left| h(x) \right| dx \right] ds \right\}^{1/2}. \nonumber
\end{equation}

Since $\int_a^\alpha \phi^2(s) |h(s)| dx$ and $\int_a^\alpha g^2(x) |h(x)| dx$ are both finite, we conclude that $|y|$ is bounded by a linear combination of $\phi$ and $g$ and hence $y$ is in $L^2(a, \infty)$. If (ii) holds, the proof is the same up to (10). In this case (10) becomes
\begin{equation}
|y(t)| \leq \sum_{i=1}^{n} \left[ C_i + \alpha_i \right] |y_i(t)| \nonumber
\end{equation}
\begin{equation}
+ \sum_{i=1}^{n} |y_i(t)| \left( \int_a^t \left| z_i h \int_a^s y^2 \right| ds. \right)^{1/2}. \nonumber
\end{equation}

Letting $D_i = \left( \int_a^\alpha |z_i|^2 h^2 \right)^{1/2}$ and proceeding as before (11) gets replaced by
\begin{equation}
|y(t)| \leq \phi(t) + g(t) \left\{ \int_a^t 2\phi^2(s) dx \right\}^{1/2}, \nonumber
\end{equation}
and the conclusion follows similarly.

4. Remarks. (1) Theorem 1 applied to the linear equation $Ly = hy$ yields: If $S \cup S^+ \subset L^2(a, \infty)$ and either $|h|^{1/2} \in L^2(a, \infty)$ for all $y$ in $S \cup S^+$ or $h \in L^2(a, \infty)$ for all $s \in S^+$, then all solutions of $Ly = hy$ are in $L^2(a, \infty)$.

(2) The hypothesis on $h$ is clearly satisfied if either $h$ is bounded or $h \in L^1(a, \infty)$.
(3) Theorem 1 holds for general quasi-differential operators such as \( l \) and \( l^+ \) studied in [11]. Equivalent versions of these were considered by Shinn [8] and include, in particular, those used in [7] and [1]. See also J. H. Barrett [2] and many others. The argument given above can be readily extended to \( l \) and \( l^+ \) without any major modifications. In particular, Theorem 1 applies to operators of the type

\[
Ly = (p_n y^n)^n + (p_{n-1} y^{n-1})^{n-1} + \cdots + p_0 y
\]

which include the Sturm operator \((p_1 y')' + p_0 y\).

(4) Condition (9) is essentially the same as the one used by Bradley [4].

BIBLIOGRAPHY

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