PSEUDO-UNIFORM CONVEXITY OF $H^1$ IN SEVERAL VARIABLES

LAURENCE D. HOFFMANN

Abstract. A convergence theorem of D. J. Newman for the Hardy space $H^1$ is generalized to several complex variables. Specifically, in both $H^1$ of the polydisc and $H^1$ of the ball, weak convergence, together with convergence of norms, is shown to imply norm convergence. As in Newman’s work, approximation of $L^1$ by $H^1$ is also considered. It is shown that every function in $L^1$ of the torus, (or in $L^1$ of the boundary of the ball), has a best $H^1$-approximation which, in several variables, need not be unique.

D. J. Newman [4] has shown that $H^1$ of the unit disc, while not uniformly convex, does have the following properties:

(i) Weak convergence, together with convergence of norms, implies norm convergence in $H^1$.

(ii) If the distance between $k \in L^1$ and a sequence of $H^1$-functions tends to $d$, ($d =$distance between $k$ and $H^1$), then the sequence converges in norm to the unique best $H^1$-approximation of $k$.

In this paper, (i) will be generalized to both the unit polydisc and unit ball in several variables. In fact, as in one variable, a somewhat stronger result will be obtained. On the other hand, examples will be given to show that (ii) does not generalize to these settings.

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Convergence theorem. Let $U^N$ denote the unit polydisc in the space of $N$ complex variables. The distinguished boundary of $U^N$ is the torus, $T^N$. The $H^1$-norm in $U^N$ is defined by

$$||f||_{1,N} = \sup_{0 < r < 1} \int_{T^N} |f(rw)| \, dm_N(w),$$

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1 I am informed by the referee that actually this result predates Newman’s paper. According to a paper by V. P. Havin [3], the theorem was first proved for the unit disc by S. Warschawski in 1930 and subsequently generalized to certain multiply connected regions by G. Ts. Tumarkin.

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where $dm_N$ denotes the Haar measure of $T^N$. $H^1(U^N)$ consists of those holomorphic functions on $U^N$ whose $H^1$-norm is finite.

If $f$ is any function on $U^N$, and $w \in T^N$, the "slice function" $f_w$ is defined on $U^1$ by $f_w(\lambda) = f(\lambda w)$ ($\lambda \in U^1$). If $h \in H^1(U^N)$, then $h_w \in H^1(U^1)$ for almost all $w \in T^N$, and, as in [6, Lemma 3.3.2], the invariance of the measure $dm_N$ implies

$$\|h\|_{1,N} = \int_{T^N} \|h_w\|_{1,1}dm_N(w).$$

**Theorem 1.** Suppose $f, f_n \in H^1(U^N)$ with

(i) $f_n \to f$ uniformly on compact subsets of $U^N$, and

(ii) $\|f\|_{1,N} \to \|f\|_{1,N}$.

Then $\|f_n - f\|_{1,N} \to 0$.

In proving this theorem for $N = 1$, Newman used a factorization of $H^1$-functions involving Blaschke products. This technique is not applicable when $N > 1$; nor is the more recent proof of C. N. Kellogg [2] in which functions in $H^1$ are expressed as products of $H^2$-functions. The proof of the theorem for $N > 1$ given here applies the one-variable result to the slice functions $f_n, w$.

**Lemma.** Suppose $\phi^0 \equiv 0$, $\phi = \lim \inf \phi_n$, $\phi$ and $\phi_n \in L^1$, $\psi \leq \phi$, and $\lim \sup \phi_n \leq \int \psi$. Then, $\phi = \psi$ a.e., and there exists $n_j \to \infty$ such that $\phi_n \to \phi$ a.e.

**Proof of Lemma.** Fatou's lemma gives the first inequality in

$$\int \phi \leq \lim \inf \int \phi_n \leq \lim \sup \int \phi_n \leq \int \psi \leq \int \phi.$$

It follows that $\phi = \psi$ a.e. and that

(1) \[ \lim \int \phi_n = \int \phi. \]

If $g_n = \inf \{\phi_n, \phi_{n+1}, \cdots\}$, the monotone convergence theorem gives

(2) \[ \lim \int g_n = \int \phi. \]

Since $g_n \leq \phi_n$, (1) and (2) imply $\int |g_n - \phi_n| \to 0$; hence, $(g_{n_j} - \phi_{n_j}) \to 0$ a.e. for some sequence $n_j \to \infty$. But $g_n \to \phi$ a.e., so $\phi_{n_j} \to \phi$ a.e.

**Proof of Theorem 1.** For $w \in T^N$, define

$$\phi_n(w) = \|f_n, w\|_{1,1}, \text{ and } \psi(w) = \|f, w\|_{1,1}.$$
For $0 < r < 1$, hypothesis (i) gives
\[ \int_{T^1} |f_n(r\lambda)| \, dm_1(\lambda) = \lim_{n \to \infty} \int_{T^1} |f_n(u(r\lambda))| \, dm_1(\lambda) \leq \liminf_{n \to \infty} \phi_n(w), \]
so that
\[ (3) \quad \psi(w) \leq \liminf_{n \to \infty} \phi_n(w). \]
Hypothesis (ii) says that
\[ (4) \quad \lim_{n \to \infty} \int_{T^N} \phi_n = \int_{T^N} \psi. \]

By the lemma, (3) and (4) imply that every sequence $S_1$ of positive integers contains a subsequence $S_2$ such that for almost all $w$, $\phi_n(w) \to \psi(w)$ as $n \to \infty$ in $S_2$. For such $w$, Newman’s theorem asserts that
\[ (5) \quad \|f_n,w - f_w\|_{1,1} \to 0 \quad \text{as } n \to \infty \quad \text{in } S_2. \]

We must show that
\[ (6) \quad \int_{T^N} \|f_n,u - f_u\|_{1,1} dm_N(w) \to 0 \quad \text{as } n \to \infty \quad \text{in } S_2. \]

If $T^N = A \cup B$, this integral is majorized by
\[ (7) \quad \int_B \|f_n,u - f_u\|_{1,1} dm_N(w) + \int_A \phi_n(u) dm_N(w) + \int_A \psi(u) dm_N(w). \]

By Egoroff's theorem and (5), $A$ and $B$ can be so chosen that $\|f_n,u - f_u\|_{1,1} \to 0$ uniformly on $B$ as $n \to \infty$ in $S_2$, and so that $\int_A \psi < \epsilon$. It follows that $\phi_n \to \psi$ uniformly on $B$, and hence that $\int_A \phi_n < \epsilon$ for large $n \in S_2$. Hence (7) tends to zero as $n \to \infty$ in $S_2$. Thus every sequence $S_1$ contains a subsequence $S_2$ for which (6) holds, i.e., for which $\|f_n - f\|_{1,1} \to 0$. This completes the proof.

With only minor notational changes, this proof generalizes Newman’s theorem to the unit ball $B^N$. In particular, if $d\nu_N$ denotes the normalized, orthogonally invariant measure on $\partial B^N$, the $H^1$-norm for $B^N$ is defined by
\[ \|f\|_1 = \sup_{0 < r < 1} \int_{\partial B^N} |f(rw)| \, d\nu_N(w). \]

If $f$ is a function on $B^N$, and $w \in \partial B^N$, the slice function $f_w$ is defined, as before, by $f_w(\lambda) = f(\lambda w)$ ($\lambda \in U^1$). Moreover, the invariance of the measure $d\nu_N$ implies the basic equality.
Remark. As defined here, $H^1(U^2)$ can be identified with a closed subspace of $L^1(T^2)$: namely, the class of all $L^1$-functions whose Fourier transform vanishes outside the set of lattice points in the first quadrant of the plane. In [2], Kellogg shows that Newman’s theorem does not generalize to the space of all functions in $L^1(T^2)$ whose Fourier transform vanishes on a certain half plane.

Best-approximation problem. If $h \in H^1(U^N)$, the radial limits $h^*(w) = \lim_{r \to 1} h(rw)$ exist for almost all $w \in T^N$. Moreover, $h^* \in L^1(T^N)$ and

$$h(z) = \int_{T^N} P(z, w)h^*(w) \, dm_N(w) \quad (z \in U^N),$$

where $P(z, w)$ is the Poisson kernel in $U^N$. A discussion of these matters appears in [6].

Let $\|f\|_{1,N}$ denote the $L^1$-norm of $f \in L^1(T^N)$. Then,

$$\|f\|_{1,N} = \int_{U^N} \|f_w\|_{1,1} \, dm_N(w),$$

where now, $f_w(\lambda) = f(\lambda w)$ for $\lambda \in T^1$.

**Theorem 2.** If $k \in L^1(T^N)$, there exists a function $h_0 \in H^1(U^N)$ for which

$$\|k - h_0\|_{1,N} = \inf \{ \|k - h^*\|_{1,N} : h \in H^1(U^N) \}.$$

For $N = 1$, Theorem 2 is derived easily from a result of Rogosinski and Shapiro [5, Theorem 8, p. 303]. A similar argument could be given when $N > 1$. However, the form of the proof given here seems somewhat more conducive to further generalization.

**Proof of Theorem 2.** Let $d = \inf \{ \|k - h^*\|_{1,N} : h \in H^1(U^N) \}$. There exist functions $h_n \in H^1(U^N)$, and a complex measure $d\mu$ on $T^N$ such that $\|k - h_n^*\|_{1,N} \to d$, and

$$d\mu = \text{weak-star limit of } h_n^* \, dm_N.$$

For $z \in U^N$, define $h_0(z) = \int_{T^N} P(z, w) \, d\mu(w)$. It follows from (8) that the Fourier coefficients of $h_n^*$ converge to those of $d\mu$. Hence $h_0$ is holomorphic. And since $\|h_0\|_{1,N} \leq \|\mu\|$, $h_0$ is in $H^1(U^N)$. Hence $d\mu = h_0^* \, dm_N$, and (8) implies $\|k - h_0^*\|_{1,N} \leq \lim \|k - h_n^*\|_{1,N} = d$. This completes the proof.
Theorem 2 holds with $U^N$ and $T^N$ replaced by $B^N$ and $\partial B^N$ respectively. The proof is identical to that just given, with one exception; we can no longer use Fourier coefficients to show that $h_0$ is holomorphic. Instead, we note that the Poisson integral representation of $h_n$, together with the boundedness of $\{||h_n^*||_{1,N}\}$, implies uniform boundedness of $\{h_n\}$ on compact sets. In addition, weak-star convergence of $h_n^*d\nu$ to $d\mu$ gives pointwise convergence of $h_n$ to $h_0$. It follows that $h_n \rightarrow h_0$ uniformly on compact subsets of $B^N$. (Lars Gårding and Lars Hörmander develop the relevant properties of $H^1(B^N)$ in [1].)

**Theorem 3.** If $N>1$, there exists a function $k\in L^1(T^N)$ having infinitely many best $H^1$-approximations, (each of them constant).

**Proof of Theorem 3.** Suppose $k\in L^1(T^N)$ and

(9) \[ k(\omega) = k(\lambda \omega), \quad \text{for all } \lambda \in T^1. \]

Let $h\in H^1(U^N)$. For almost all $\omega$, the slice function $h_\omega$ is in $H^1(U^1)$, while by (9), each $k_\omega$ is constant. It follows that

\[ ||k_\omega - h(0)||_{1,1} = |k_\omega(\lambda) - h(0)| = \left| \int_{T^1} (k_\omega(\lambda) - h_\omega^*(\lambda))dm_1(\lambda) \right|, \]

and consequently,

(10) \[ ||k - h(0)||_{1,N} \leq \int_{T^N} \int_{T^1} |k_\omega(\lambda) - h_\omega^*(\lambda)| = ||k - h^*||_{1,N}. \]

Since equality holds in (10) only when $k$ is constant, we conclude that every best $H^1$-approximation of $k$ is constant.

To construct $k$ with infinitely many best approximations, choose a subset $E$ of $T^1$ with $m_1(E) = 1/2$, and let $A$ be the set of all points $(w_1, \ldots, w_N)$ in $T^N$, for which $w_1w_2 \cdots w_{N-1}(\tilde{w}_N)^{N-1}$ is in $E$. Define $k(w)$ to be 1 if $w \in A$ and $-1$ if $w \in T^N - A$. Notice that $k$ satisfies (9), and that $m_N(A) = 1/2$. It follows that the best $H^1$-approximations of $k$ are precisely the real constants, $c$, for which $-1 \leq c \leq 1$.

Except for the definition of $k$, this proof applies without change to the analogous theorem for the ball. To define an appropriate function $k \in L^1(\partial B^N)$, let $E$ be the set of all $(a_1, \ldots, a_N)$ in $R^N$ for which $\sum a_i^2 = 1$, $a_i \geq 0$, and $a_1 \geq a_2$. Let $A$ consist of those points $(a_1\lambda_1, \ldots, a_N\lambda_N)$ in $\partial B^N$ with $(a_1, \ldots, a_N)$ in $E$, and $(\lambda_1, \ldots, \lambda_N)$ in $T^N$. Since the measure of $A$ is $1/2$, the function $k$ which is 1 on $A$ and $-1$ on $\partial B^N - A$ has the required property.
REFERENCES


UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706