A 2-SPHERE OF VERTICAL ORDER 5 BOUNDS A 3-CELL[1]

L. D. LOVELAND

ABSTRACT. A subset X of E^3 is said to have *vertical order* n if no vertical line contains more than n points of X. We prove that each 2-sphere in E^3 which has vertical order 5 bounds a 3-cell.

Jensen and Loveland [7] proved that a 2-sphere in E^3 having vertical order 3 is tame. It was pointed out there that 3 is the largest number with this property since the Alexander Horned Sphere [1] and a wild sphere given by Fox and Artin [6] can each be described in E^3 so as to have vertical order 4. In both of these examples the wild spheres are tame from their interiors; that is, each bounds a 3-cell in E^3. In this note we prove that this is the case for each 2-sphere of vertical order 5. Again modifications of the above mentioned examples show that 5 is the largest number with this property. Our proof depends upon recent developments in $*$-taming set theory by J. W. Cannon [5] and makes use of the main result in [7].

A *crumpled cube* in E^3 is the union of a 2-sphere in E^3 with its bounded complementary domain in E^3. For our purposes in this note, a *$*$-taming set* for crumpled cubes in E^3 is a closed set X having the property that if C is a crumpled cube such that $X \subset E^3 - \operatorname{Int} C$ and $\operatorname{Bd} C$ is locally tame modulo X, then C is a 3-cell. The basic properties of $*$-taming sets were developed by J. W. Cannon and can be found in [4]. Another excellent reference for definitions and results not given here is the survey article by Burgess and Cannon [3] which also contains some of the early results on $*$-taming set theory.

THEOREM. *A 2-sphere in E^3 having vertical order 5 bounds a 3-cell in E^3.*

PROOF. Let C be a crumpled cube such that its boundary S is a 2-sphere having vertical order 5, and let P be a vertical plane. We select a coordinate system such that P is the $z-y$ coordinate plane, and we denote the plane $\{(x, y, z) \mid x = r\}$ by $P(r)$. We let $C^* = E^3 - \operatorname{Int} C$, and we define $X(r) = C^* \cap P(r)$, for each real number r. If t is a posi-

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tive number, then $X^t(r)$ denotes the union of all the components of $X(r)$ having diameter no less than t. It follows from Theorem 1 of [5] that the closed set $X^t = \bigcup_{r\in R} X^t(r)$ is a $*$-taming set if $t>0$. We shall construct a countable collection $\{D_i\}$ of tame disks on S such that each point component of each $X(r)$ lies in one of these disks. Then we shall show that

$$C^* = \bigcup_{i=1}^{\infty} (D_i \cup X^{1/i}).$$

Since each $X^{1/i}$ is a $*$-taming set and a tame disk is also a $*$-taming set [4], it follows from [4] that their closed countable union C^* is also a $*$-taming set (these results are also stated in §8.2 of [3]). Once all this is accomplished it follows directly from the definition of $*$-taming sets that C is a 3-cell.

We now indicate how to cover each point component $\{p\}$ of $X(r)$ with the interior of a tame disk on S. If we let $L(p)$ denote the vertical line through p we see that p is the common endpoint of two open intervals $I(a)$ and $I(b)$ in $L(p)\cap\text{Int } C$. This means that there must exist points m and n of $L(p)\cap S$, above and below p respectively, such that $L(p)$ pierces S at both m and n. We select two disjoint disks M and N in S such that $L(p)\cap(M\cup N)=\{m, n\}$. Since $L(p)$ pierces M at m it is clear that any vertical line sufficiently close to $L(p)$ will also intersect M. Using this property on both M and N we are able to select a disk D' in S such that $p\in\text{Int } D'$, $D'\cap(M\cup N)=\emptyset$, $D'\cap L(p) = \{p\}$, and D' has small enough diameter to insure that each vertical line intersecting it must also intersect both M and N. Thus D' has vertical order 3. In the next paragraph we indicate how to obtain a 2-sphere S' and two disks D and F in S' such that $p\in\text{Int } D\subset D\subset \text{Int } F\subset D'$ and such that no vertical line intersecting D also intersects $S'-F$. These properties together with Lemma 1, which follows this proof, insure that D is a tame disk.

We may assume that there is a disk E such that $p\in\text{Int } E$, $\text{Bd } E = \text{Bd } D'$, $E\cap L(p)=\{p\}$, $E\cap(S-D')=\emptyset$, and E is locally polyhedral modulo $\text{Bd } E\cup\{p\}$ [2]. Let K be the 2-sphere $(S-D')\cup E$ and note that $I(a)$ lies in Int K. Since $I(a)\cup\{p\}$ is a tame arc, K is tamely arcwise accessible from its interior at p. Thus p is a piercing point of $K^*=E^3-\text{Int } K$ [10, Theorem 3], and it follows that K is locally tame from Ext K at p (see for example [9, Theorem 2] and [8, Theorem 14], or alternatively use the fact that $I(a)\cup\{p\}$ is a $*$-taming set [4, Theorem 4]). Thus E may be pushed slightly into Ext K near p to obtain a disk E' such that $\text{Bd } E'=\text{Bd } D'$, E' is locally polyhedral in its interior, and $L(p)\cap E'=\emptyset$. We choose a disk

F in D' such that $p \in \operatorname{Int} F$ and no vertical line intersecting F intersects E'. Now we choose D to be a disk in Int F such that $p \in \operatorname{Int} D$ and no vertical line through D intersects $D'-F$. Let V be the set of all vertical lines intersecting D and adjust the annulus $A = D'-F$ in $E^3 - V$ so that it becomes locally polyhedral [2] and in general position with E'. Then the component G of $(A \cup F) - E'$ which contains F is a punctured disk. The holes in G can be filled with disks in or near E' to obtain the desired 2-sphere S'.

Thus each component of each $X(r)$ lies in the interior of a tame disk in S. Of course the union of these point components must then be covered by some countable collection $\{D_i\}$ of such tame disks.

We show now that $C^* = \bigcup_{i=1}^{\infty} (X^{1/i} \cup D_i)$. Let $y \in C^*$. If $y \notin S$, then y lies in some vertical interval in C^* of length greater than $1/i$, for some i, so that $y \in X^{1/i}$. If $y \in S$ and y does not belong to any $X^{1/i}$, then y is a point component of some $X(r)$. This means that y lies in some tame disk D_i. This completes the proof since we have exhibited C^* as the countable union of $*$-taming sets.

LEMMA 1. *Suppose that F is a disk having vertical order* 3 *which lies on a* 2*-sphere S in E^3. If D is a subdisk of F such that no vertical line through D intersects $S-F$, then S is locally tame at each point of* Int D.

PROOF. Let $p \in$ Int D, and denote the vertical line containing $x \in E^3$ by $L(x)$. If $L(p)$ intersects Int S it is easy to see that S is locally tame at p. This is because p would then lie in the interior of a subdisk of D which has vertical order 1. Such a disk could be bicollared (or locally spanned [3]) at its interior points using short vertical intervals. Thus we may assume that $L(p) \cap \operatorname{Int} S = \emptyset$ which means that $L(p)$ does not pierce S at any point. We let H be a right circular cylindrical surface (of finite height) having $L(p)$ as its axis such that p is the only point of $S \cap L(p)$ in the bounded component Int H of $E^3 - H$, Bd $D \subset$ Ext H, and the top $D(t)$ and bottom $D(B)$ of H are horizontal disks in Ext S. We shall first show that the closure of each component X of $H \cap \operatorname{Int} S$ is a disk.

We let $K = \operatorname{Bd} X$, and we note that if a vertical line $L(x)$ intersects X, then $L(x) \cap K$ consists of two points U_x and V_x. In this case we use U_x to denote the point with larger vertical coordinate, and we define

$$U = \{U_x \mid L(x) \cap X \neq \emptyset\} \quad \text{and} \quad V = \{V_x \mid L(x) \cap X \neq \emptyset\}.$$

Now $U \cup V \subset K$, and we shall show that U and V are each open arcs. First we show that U (and hence V) is arcwise connected. Let u_1 and u_2 be two points of U. By the definition of U there exist two vertical intervals I_1 and I_2 such that, for each $i=1, 2$, Int $I_i \subset X$, u_i is the upper endpoint of I_i, and the lower endpoint of I_i is a point of V. Since X is arcwise connected there is an arc B from a point of Int I_1 to a point of Int I_2 which intersects $I_1 \cup I_2$ only at its endpoints and which lies in X. Let

$$X' = \{x \in X \mid L(x) \cap B \neq \emptyset\}, \qquad U' = \{x \in U \mid L(x) \cap B \neq \emptyset\},$$
$$V' = \{x \in V \mid L(x) \cap B \neq \emptyset\},$$

and let M be the closed set $U' \cup V' \cup I_1 \cup I_2$. The following facts are easily verified: $H-M$ is the union of the disjoint connected open sets X' and $H-(M\cup X')$; every point of M is arcwise accessible from both X' and $H-(M\cup X')$. These facts insure that M is a simple closed curve [11, p. 233], so it follows that U' is an arc from u_1 to u_2. We now know that U is a connected 1-manifold.

Suppose U is a simple closed curve. Since U has vertical order 1 it is clear that U cannot bound a disk in the annulus $H'=H-(D(B) \cup D(T))$. Since $L(p)$ is the vertical axis for H' it follows that $L(p)$ must pierce the disk in F bounded by U. But this a is contradiction as $L(p)$ fails to pierce S at any point. Since each point of U lies in an open arc in U it is now clear that both U and V are open arcs.

Now $\mathrm{cl}(U)$ is connected, and since $\mathrm{cl}(U)$ has vertical order 3 we see that $\mathrm{cl}(U)$ is an arc. Also $\mathrm{cl}(V)$ is an arc lying vertically below $\mathrm{cl}(U)$. Since X is a simply connected open set it follows that its boundary K is connected. Because the connected set K lies in $\mathrm{cl}(U) \cup \mathrm{cl}(V)$ and contains both U and V we see that K is a simple closed curve.

Thus the closure of each component X of $H \cap \mathrm{Int}\, S$ is a disk whose boundary lies in $F \cap H$ and whose interior lies in $H \cap \mathrm{Int}\, S$. There are at most a countable number of these spanning disks for C, and we denote them by E_i $(i=1, 2, 3, \cdots)$. Suppose that some subsequence of $\{E_i\}$ converges to a set G. Since F has vertical order 3 we see that G lies in a vertical line; and, for the same reason, the connected subset G of $H \cap F$ must consist of a single point. It follows that $\{E_i\}$ is a null sequence of disks, and it can be proved from this fact that the closure of each nondegenerate component of $C - \bigcup E_i$ is a crumpled cube. Let N be the component of Int $C - \bigcup E_i$ which has p in its 2-sphere boundary S_1. We shall show that $N \subset H \cup \mathrm{Int}\, H$.

Suppose there is a point $q \in N \cap \mathrm{Ext}\, H$, and let A be an arc from p to q such that $A - \{p\} \subset N \subset \mathrm{Int}\, S$. Assuming, as we may, that A is in general position with respect to H, we see that A must pierce H at some point x. Since $x \in \mathrm{Int}\, S$, x must lie in the interior of some disk

E_i which separates Int C. The fact that A pierces E_i at x implies that A intersects at least two different components of Int $C - \bigcup E_i$. This produces the contradiction that A is not a subset of N.

We now have a 2-sphere S_1 containing p such that $\mathrm{cl}(S_1 - H)$ has vertical order 3. But the interiors of each of the disks E_i whose boundaries lie in $\mathrm{cl}(S_1 - H)$ can be pushed slightly into Ext H so as to obtain disks F_i having vertical order 3 whose boundaries lie in $S_1 \cap H$. We let $S' = [\mathrm{cl}\ (S_1 - H) \cup (\bigcup F_i)]$ and we note that S' is a tame 2-sphere [**7**]. Thus S is locally tame at p and the result follows.

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Utah State University, Logan, Utah 84321