A 2-Sphere of Vertical Order 5 BOUNDS A 3-CELL¹

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Abstract. A subset X of \( E^3 \) is said to have vertical order n if no vertical line contains more than n points of X. We prove that each 2-sphere in \( E^3 \) which has vertical order 5 bounds a 3-cell.

Jensen and Loveland [7] proved that a 2-sphere in \( E^3 \) having vertical order 3 is tame. It was pointed out there that 3 is the largest number with this property since the Alexander Horned Sphere [1] and a wild sphere given by Fox and Artin [6] can each be described in \( E^3 \) so as to have vertical order 4. In both of these examples the wild spheres are tame from their interiors; that is, each bounds a 3-cell in \( E^3 \). In this note we prove that this is the case for each 2-sphere of vertical order 5. Again modifications of the above mentioned examples show that 5 is the largest number with this property. Our proof depends upon recent developments in \(*\)-taming set theory by J. W. Cannon [5] and makes use of the main result in [7].

A crumpled cube in \( E^3 \) is the union of a 2-sphere in \( E^3 \) with its bounded complementary domain in \( E^3 \). For our purposes in this note, a \(*\)-taming set for crumpled cubes in \( E^3 \) is a closed set \( X \) having the property that if \( C \) is a crumpled cube such that \( X \subseteq E^3 - \text{Int } C \) and Bd \( C \) is locally tame modulo \( X \), then \( C \) is a 3-cell. The basic properties of \(*\)-taming sets were developed by J. W. Cannon and can be found in [4]. Another excellent reference for definitions and results not given here is the survey article by Burgess and Cannon [3] which also contains some of the early results on \(*\)-taming set theory.

Theorem. A 2-sphere in \( E^3 \) having vertical order 5 bounds a 3-cell in \( E^3 \).

Proof. Let \( C \) be a crumpled cube such that its boundary \( S \) is a 2-sphere having vertical order 5, and let \( P \) be a vertical plane. We select a coordinate system such that \( P \) is the \( z-y \) coordinate plane, and we denote the plane \( \{ (x, y, z) \mid x = r \} \) by \( P(r) \). We let \( C^* = E^3 - \text{Int } C \), and we define \( X(r) = C^* \cap P(r) \), for each real number \( r \). If \( t \) is a posi-

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tive number, then \(X^t(r)\) denotes the union of all the components of \(X(r)\) having diameter no less than \(t\). It follows from Theorem 1 of [5] that the closed set \(X^t = \bigcup_{r \in R} X^t(r)\) is a \(*\)-taming set if \(t > 0\). We shall construct a countable collection \(\{D_i\}\) of tame disks on \(S\) such that each point component of each \(X^t(r)\) lies in one of these disks. Then we shall show that

\[
C^* = \bigcup_{i=1}^{\infty} (D_i \cup X^t(i)).
\]

Since each \(X^t(i)\) is a \(*\)-taming set and a tame disk is also a \(*\)-taming set [4], it follows from [4] that their closed countable union \(C^*\) is also a \(*\)-taming set (these results are also stated in §8.2 of [3]). Once all this is accomplished it follows directly from the definition of \(*\)-taming sets that \(C\) is a 3-cell.

We now indicate how to cover each point component \(\{p\}\) of \(X(r)\) with the interior of a tame disk on \(S\). If we let \(L(p)\) denote the vertical line through \(p\) we see that \(p\) is the common endpoint of two open intervals \(I(a)\) and \(I(b)\) in \(L(p) \cap \text{Int} \ C\). This means that there must exist points \(m\) and \(n\) of \(L(p) \cap S\), above and below \(p\) respectively, such that \(L(p)\) pierces \(S\) at both \(m\) and \(n\). We select two disjoint disks \(M\) and \(N\) in \(S\) such that \(L(p) \cap (M \cup N) = \{m, n\}\). Since \(L(p)\) pierces \(M\) at \(m\) it is clear that any vertical line sufficiently close to \(L(p)\) will also intersect \(M\). Using this property on both \(M\) and \(N\) we are able to select a disk \(D'\) in \(S\) such that \(p \in \text{Int} \ D', \ D' \cap (M \cup N) = \emptyset, \ D' \cap L(p) = \{p\}\), and \(D'\) has small enough diameter to insure that each vertical line intersecting it must also intersect both \(M\) and \(N\). Thus \(D'\) has vertical order \(3\). In the next paragraph we indicate how to obtain a 2-sphere \(S'\) and two disks \(D\) and \(F\) in \(S'\) such that \(p \in \text{Int} \ D \subset \text{Int} \ D' \subset \text{Int} \ F \subset D'\) and such that no vertical line intersecting \(D\) also intersects \(S' - F\). These properties together with Lemma 1, which follows this proof, insure that \(D\) is a tame disk.

We may assume that there is a disk \(E\) such that \(p \in \text{Int} \ E, \ \text{Bd} \ E = \text{Bd} \ D', \ E \cap L(p) = \{p\}, \ E \cap (S - D') = \emptyset, \) and \(E\) is locally polyhedral modulo \(\text{Bd} \ E \cup \{p\}\) [2]. Let \(K\) be the 2-sphere \((S - D') \cup E\) and note that \(I(a)\) lies in \(\text{Int} \ K\). Since \(I(a) \cup \{p\}\) is a tame arc, \(K\) is tamely arcwise accessible from its interior at \(p\). Thus \(p\) is a piercing point of \(K^* = E^3 - \text{Int} \ K\) [10, Theorem 3], and it follows that \(K\) is locally tame from \(\text{Ext} \ K\) at \(p\) (see for example [9, Theorem 2] and [8, Theorem 14]), or alternatively use the fact that \(I(a) \cup \{p\}\) is a \(*\)-taming set [4, Theorem 4]). Thus \(E\) may be pushed slightly into \(\text{Ext} \ K\) near \(p\) to obtain a disk \(E'\) such that \(\text{Bd} \ E' = \text{Bd} \ D', \ E'\) is locally polyhedral in its interior, and \(L(p) \cap E' = \emptyset\). We choose a disk
F in D' such that p ∈ Int F and no vertical line intersecting F intersects E'. Now we choose D to be a disk in Int F such that p ∈ Int D and no vertical line through D intersects D' − F. Let V be the set of all vertical lines intersecting D and adjust the annulus A = D' − F in E^3 − V so that it becomes locally polyhedral [2] and in general position with E'. Then the component G of (A ∪ F) − E' which contains F is a punctured disk. The holes in G can be filled with disks in or near E' to obtain the desired 2-sphere S'.

Thus each component of each X(r) lies in the interior of a tame disk in S. Of course the union of these point components must then be covered by some countable collection \{D_i\} of such tame disks.

We show now that C* = \bigcup_{i=1}^{\infty} (X^{1/i} \cup D_i). Let y ∈ C*. If y ∈ S, then y lies in some vertical interval in C* of length greater than 1/i, for some i, so that y ∈ X^{1/i}. If y ∈ S and y does not belong to any X^{1/i}, then y is a point component of some X(r). This means that y lies in some tame disk D_i. This completes the proof since we have exhibited C* as the countable union of *-taming sets.

**Lemma 1.** Suppose that F is a disk having vertical order 3 which lies on a 2-sphere S in E^3. If D is a subdisk of F such that no vertical line through D intersects S − F, then S is locally tame at each point of Int D.

**Proof.** Let p ∈ Int D, and denote the vertical line containing x ∈ E^3 by L(x). If L(p) intersects Int S it is easy to see that S is locally tame at p. This is because p would then lie in the interior of a subdisk of D which has vertical order 1. Such a disk could be bicol-lared (or locally spanned [3]) at its interior points using short vertical intervals. Thus we may assume that L(p) \∩ Int S = ∅ which means that L(p) does not pierce S at any point. We let H be a right circular cylindrical surface (of finite height) having L(p) as its axis such that p is the only point of S \∩ L(p) in the bounded component Int H of E^3 − H, Bd D ⊂ Ext H, and the top D(t) and bottom D(B) of H are horizontal disks in Ext S. We shall first show that the closure of each component X of H \∩ Int S is a disk.

We let K = Bd X, and we note that if a vertical line L(x) intersects X, then L(x) \∩ K consists of two points U_x and V_x. In this case we use U_x to denote the point with larger vertical coordinate, and we define

\[ U = \{ U_x \mid L(x) \cap X \neq \emptyset \} \quad \text{and} \quad V = \{ V_x \mid L(x) \cap X \neq \emptyset \}. \]

Now U ∪ V ⊂ K, and we shall show that U and V are each open arcs. First we show that U (and hence V) is arcwise connected. Let u_1 and u_2 be two points of U. By the definition of U there exist two
vertical intervals $I_1$ and $I_2$ such that, for each $i = 1, 2$, $\text{Int } I_i \subset X$, $u_i$ is the upper endpoint of $I_i$, and the lower endpoint of $I_i$ is a point of $V$. Since $X$ is arcwise connected there is an arc $B$ from a point of $\text{Int } I_1$ to a point of $\text{Int } I_2$ which intersects $I_1 \cup I_2$ only at its endpoints and which lies in $X$. Let

$$X' = \{x \in X \mid L(x) \cap B \neq \emptyset\}, \quad U' = \{x \in U \mid L(x) \cap B \neq \emptyset\},$$

$$V' = \{x \in V \mid L(x) \cap B \neq \emptyset\},$$

and let $M$ be the closed set $U' \cup V' \cup I_1 \cup I_2$. The following facts are easily verified: $H - M$ is the union of the disjoint connected open sets $X'$ and $H - (M \cup X')$; every point of $M$ is arcwise accessible from both $X'$ and $H - (M \cup X')$. These facts insure that $M$ is a simple closed curve [11, p. 233], so it follows that $U'$ is an arc from $u_1$ to $u_2$. We now know that $U$ is a connected 1-manifold.

Suppose $U$ is a simple closed curve. Since $U$ has vertical order 1 it is clear that $U$ cannot bound a disk in the annulus $H' = H - (D(B) \cup D(T))$. Since $L(p)$ is the vertical axis for $H'$ it follows that $L(p)$ must pierce the disk in $F$ bounded by $U$. But this a is contradiction as $L(p)$ fails to pierce $S$ at any point. Since each point of $U$ lies in an open arc in $U$ it is now clear that both $U$ and $V$ are open arcs.

Now $\text{cl}(U)$ is connected, and since $\text{cl}(U)$ has vertical order 3 we see that $\text{cl}(U)$ is an arc. Also $\text{cl}(V)$ is an arc lying vertically below $\text{cl}(U)$. Since $X$ is a simply connected open set it follows that its boundary $K$ is connected. Because the connected set $K$ lies in $\text{cl}(U) \cup \text{cl}(V)$ and contains both $U$ and $V$ we see that $K$ is a simple closed curve.

Thus the closure of each component $X$ of $H \cap \text{Int } S$ is a disk whose boundary lies in $F \cap H$ and whose interior lies in $H \cap \text{Int } S$. There are at most a countable number of these spanning disks for $C$, and we denote them by $E_i$ $(i = 1, 2, 3, \cdots)$. Suppose that some subsequence of $\{E_i\}$ converges to a set $G$. Since $F$ has vertical order 3 we see that $G$ lies in a vertical line; and, for the same reason, the connected subset $G$ of $H \cap F$ must consist of a single point. It follows that $\{E_i\}$ is a null sequence of disks, and it can be proved from this fact that the closure of each nondegenerate component of $C - U E_i$ is a crumpled cube. Let $N$ be the component of $\text{Int } C - U E_i$ which has $p$ in its 2-sphere boundary $S_i$. We shall show that $N \subset H \cup \text{Int } H$.

Suppose there is a point $q \in N \cap \text{Ext } H$, and let $A$ be an arc from $p$ to $q$ such that $A - \{p\} \subset N \subset \text{Int } S$. Assuming, as we may, that $A$ is in general position with respect to $H$, we see that $A$ must pierce $H$ at some point $x$. Since $x \in \text{Int } S$, $x$ must lie in the interior of some disk.
E, which separates Int C. The fact that A pierces E, at x implies that A intersects at least two different components of Int C — U E, This produces the contradiction that A is not a subset of N.

We now have a 2-sphere S, containing p such that cl(S, — H) has vertical order 3. But the interiors of each of the disks E, whose boundaries lie in cl(S, — H) can be pushed slightly into Ext H so as to obtain disks F, having vertical order 3 whose boundaries lie in S, ∩ H. We let S' = [cl (S, — H) \( \cup (\bigcup F_i) \)] and we note that S' is a tame 2-sphere [7]. Thus S is locally tame at p and the result follows.

References


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