A SHORT PROOF OF BERGER'S CURVATURE TENSOR ESTIMATES

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ABSTRACT. A simple proof of Berger's estimate of the curvature tensor components in terms of the pinching is given and the result is extended to nonorthonormal arguments of the curvature tensor.

Let M be a Riemannian manifold with scalar product (;) and curvature tensor $R: TM \times TM \to TM$. The sectional curvatures of M at p are given by

(1) $K(u, v) = (R(u, v)v; u) \cdot ((u; u)(v; v) - (u; v)^2)^{-1}$, for independent $u, v \in M_p$.

Put

$$(2) \quad \delta = \min_{u,v \in M_p} K(u, v), \quad \Delta = \max_{u,v \in M_p} K(u, v), \quad \overline{K} = \max(|\delta|, |\Delta|).$$

$(\Delta - \delta) \cdot \overline{K}^{-1}$ is the pinching of M.

Berger's result [1], [2] (orginally for $\delta \geqq 0$, but this is not needed as Tsagas [4] observed) is

(3) $|(R(u, v)v; w)| \leqq \frac{1}{2}(\Delta - \delta)$ *for orthonormal* $u, v, w \in M_p$,

(4) $|(R(u, v)x; w)| \leqq \frac{2}{3}(\Delta - \delta)$ *for orthonormal* $u, v, w, x \in M_p$.

The constants $\frac{1}{2}$ *and* $\frac{2}{3}$ *cannot be improved. Apply Theorem* 3.2.7 *of* [2, p. 92] *to* CP^n.

PROOF OF (3) AND (4). We use the curvature tensor identities

$$(5) \quad 4(R(u, v)v; w) = (R(u + w, v)v; u + w) - (R(u - w, v)v; u - w),$$

$$(6) \quad \begin{aligned} 6(R(u, v)x; w) = {} & (R(u, v + x)v + x; w) - (R(u, v - x)v - x; w) \\ & - (R(v, u + x)u + x; w) + (R(v, u - x)u - x; w), \end{aligned}$$

which are immediate from the linearity and symmetries of R. (5) and (6) express $(R(u, v)x; w)$ in a simpler way in terms of sectional curvatures than is given in [3, p. 93].

For orthonormal u, v, w we have $(u \pm w)^2 = 2$, hence (3) is clear from (5), (1), (2). If u, v, w, x are orthonormal then (3) applies to the four terms on the right of (6); together with $(v \pm x)^2 = (u \pm x)^2 = 2$ we have (4).

To formulate the extensions of (3) and (4) we interpret the endpoints of three unit vectors $u, v, w \in M_p$ as vertices of a spherical tri-

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angle. With the notation $(u;\ v)^2=\cos^2(u,\ v)=1-\sin^2(u,\ v)$ the spherical cosine formula reads

$$\cos(u, w) = \cos(u, v)\cdot\cos(v, w) + \sin(u, v)\cdot\sin(v, w)\cdot\cos\angle uvw.$$

THEOREM. *The following generalization of* (3) *and* (4) *holds* (*for unit vectors* $u, v, w, x\in M_p$)

$$\begin{aligned} |R(u, v)v; w)| &\leqq |\sin(u, v)\cdot\sin(v, w)| \\ &\quad\cdot\{\tfrac{1}{2}(\Delta-\delta)|\sin\angle uvw| + \overline{K}\cdot|\cos\angle uvw|\} \\ &\leqq (\overline{K}^2+\tfrac{1}{4}(\Delta-\delta)^2)^{1/2}, \end{aligned} \tag{7}$$

where $\cos\angle uvw=0$ *if* $w\perp u, v$.

$$|(R(u, v)x; w)| \leqq |\sin(u, v)\cdot\sin(x, w)|\cdot\tfrac{2}{3}(\Delta-\delta), \quad \textit{for } w\perp u, v. \tag{8}$$

$$|(R(u, v)x; w)| \leqq (\overline{K}^2+(25/36)(\Delta-\delta)^2)^{1/2}, \tag{9}$$

hence, if $\|R\|$ *denotes the norm of* R *as quadrilinear map then* $\|R\| \leqq (\overline{K}^2+(25/36)(\Delta-\delta)^2)^{1/2}$.

PROOF. Put $y=u-v(u;\ v)$, $z=w-v(w;\ v)$, then $|y|=|\sin(u,\ v)|$, $|z|=|\sin(w,\ v)|$ and $v\perp y,\ z$. Therefore we can apply (3) to the first term on the right of

$$\begin{aligned}(R(u, v)v; w) &= \left(R(y, v)v; z-\frac{y}{|y|^2}(y; z)\right) \\ &\quad+\left(R(y, v)v; \frac{y}{|y|^2}(y; z)\right)\end{aligned}$$

and get the first part of (7) since—by the spherical cosine formula—

$$|\cos(y, z)| = \frac{|(y; z)|}{|y||z|} = |\cos\angle uvw|$$

and

$$\begin{aligned}|y|\cdot\left|z-\frac{y}{|y|^2}(y; z)\right| &= |y||z||\sin(y, z)| \\ &= |\sin(u, v)\cdot\sin(v, w)\cdot\sin\angle uvw|.\end{aligned}$$

The second part of (7) follows from $|A\cdot\sin\alpha|+|B\cdot\cos\alpha| \leqq (A^2+B^2)^{1/2}$. The sin-factors in (8) are trivial since with

$$u' = (u-v(u; v))\cdot|\sin(u, v)|^{-1}, \qquad x' = (x-w(x; w))\cdot|\sin(x, w)|^{-1}$$

we have

$$(R(u, v)x; w) = |\sin(u, v)\cdot\sin(x, w)|\cdot(R(u', v)x'; w).$$

We apply (7) to the four terms on the right of (6) (with u', x' instead of u, x) and since $w \perp u'$, v, x' all the $\overline{K}\cdot$cos-terms vanish. Hence (8) follows using $(v+x')^2+(v-x')^2+(u'+x')^2+(u'-x')^2=8$. In (9) we can assume $u \perp v$, $x \perp w$ (replace u, x by u', x'). Put $y=u-x(u;x)$, $z=v-x(v;x)$, $a=u(v;x)-v(u;x)$. Note $a \perp x$ and $y(v;x)-v(u;x)=a-x(u;x)(v;x)$. Then

$$\begin{aligned}(R(u,v)x;w) &= (R(y,z)x;w) + (R(y,x(v;x))x;w) + (R(x(u;x),v)x;w)\\ &= (R(y,z)x;w) + (R(a,x)x;w).\end{aligned}$$

From (8) we have

$$\begin{aligned}|(R(y,z)x;w)| &\leqq \tfrac{2}{3}(\Delta-\delta)\,|y|\,|z|\,|\sin(y,z)|\\ &= \tfrac{2}{3}(\Delta-\delta)\,|\sin(u,x)\cdot\sin(v,x)\cdot\sin\angle uxv|,\end{aligned}$$

and from (7) we have

$$|(R(a,x)x;w)| \leqq |a|\cdot(\overline{K}^2+\tfrac{1}{4}(\Delta-\delta)^2)^{1/2}.$$

Observe $|a|^2=(u;x)^2+(v;x)^2\leqq|x|^2=1$ and—from the spherical cosine formula for the triangle u, x, v—

$$|\sin(u,x)\cdot\sin(v,x)\cdot\sin\angle uxv|^2 = 1-|a|^2.$$

Then (9) follows again from $|A\cdot\sin\alpha|+|B\cdot\cos\alpha|\leqq(A^2+B^2)^{1/2}$.

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