A SHORT PROOF OF BERGER'S CURVATURE TENSOR ESTIMATES

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Abstract. A simple proof of Berger's estimate of the curvature tensor components in terms of the pinching is given and the result is extended to nonorthonormal arguments of the curvature tensor.

Let \( M \) be a Riemannian manifold with scalar product \( ( ; ) \) and curvature tensor \( R : TM \times TM \to TM \). The sectional curvatures of \( M \) at \( p \) are given by

\[ K(u, v) = (R(u, v)v; u) \cdot ((u; u)(v; v) - (u; v)^2)^{-1}, \]

for independent \( u, v \in M_p \).

Put

\[ \delta = \min_{u, v \in M_p} K(u, v), \quad \Delta = \max_{u, v \in M_p} K(u, v), \quad K = \max( |\delta|, |\Delta| ). \]

\((\Delta - \delta) \cdot K^{-1}\) is the pinching of \( M \).

Berger's result [1], [2] (originally for \( \delta \geq 0 \), but this is not needed as Tsagas [4] observed) is

\[ (R(u, v)v; w) \leq \frac{1}{3}(\Delta - \delta) \quad \text{for orthonormal } u, v, w \in M_p, \]

\[ (R(u, v)x; w) \leq \frac{1}{3}(\Delta - \delta) \quad \text{for orthonormal } u, v, w, x \in M_p. \]

The constants \( \frac{1}{3} \) and \( \frac{1}{3} \) cannot be improved. Apply Theorem 3.2.7 of [2, p. 92] to \( CP^n \).

Proof of (3) and (4). We use the curvature tensor identities

\[ 4(R(u, v)v; w) = (R(u + w, v)v; u + w) - (R(u - w, v)v; u - w), \]

\[ 6(R(u, v)x; w) = (R(u, v + x)v + x; w) - (R(u, v - x)v - x; w) \]

\[ - (R(v, u + x)u + x; w) + (R(v, u - x)u - x; w), \]

which are immediate from the linearity and symmetries of \( R \). (5) and (6) express \( (R(u, v)x; w) \) in a simpler way in terms of sectional curvatures than is given in [3, p. 93].

For orthonormal \( u, v, w \) we have \( (u \pm w)^2 = 2 \), hence (3) is clear from (5), (1), (2). If \( u, v, w, x \) are orthonormal then (3) applies to the four terms on the right of (6); together with \( (v \pm x)^2 = (u \pm x)^2 = 2 \) we have (4).

To formulate the extensions of (3) and (4) we interpret the endpoints of three unit vectors \( u, v, w \in M_p \) as vertices of a spherical tri-
angle. With the notation \((u; v)^2 = \cos^2(u, v) = 1 - \sin^2(u, v)\) the spherical cosine formula reads

\[
\cos(u, w) = \cos(u, v) \cdot \cos(v, w) + \sin(u, v) \cdot \sin(v, w) \cdot \cos \angle uvw.
\]

**Theorem.** The following generalization of (3) and (4) holds (for unit vectors \(u, v, w, x \in \mathbb{S}_p\))

\[
|R(u, v)v; w)| \leq |\sin(u, v) \cdot \sin(v, w)| 
\cdot \left\{ \frac{1}{2}(\Delta - \delta) \cdot \sin \angle uvw + K \cdot |\cos \angle uvw| \right\}
\leq (K^2 + \frac{1}{4}(\Delta - \delta)^2)^{1/2},
\]

where \(\cos \angle uvw = 0\) if \(w \bot u, v\).

\[
| (R(u, v)x; w)| \leq |\sin(u, v) \cdot \sin(x, w)| \cdot \frac{3}{4}(\Delta - \delta), \text{ for } w \bot u, v.
\]

\[
| (R(u, v)x; w)| \leq (K^2 + (25/36)(\Delta - \delta)^2)^{1/2},
\]

**hence, if** \(||R||\) **denotes the norm of** \(R\) **as quadrilinear map then** \(||R|| \leq (K^2 + (25/36)(\Delta - \delta)^2)^{1/2}\).

**Proof.** Put \(y = u - v(u; v), z = w - v(w; v), \) then \(|y| = |\sin(u, v)|, |z| = |\sin(w, v)|\) and \(v \perp y, z\). Therefore we can apply (3) to the first term on the right of

\[
(R(u, v)v; w) = \left( R(y, v)v; z - \frac{y}{|y|^2} (y; z) \right)
\]

\[
+ \left( R(y, v)v; \frac{y}{|y|^2} (y; z) \right)
\]

and get the first part of (7) since—by the spherical cosine formula—

\[
|\cos(y, z)| = \frac{|y; z|}{|y| \cdot |z|} = |\cos \angle uvw|
\]

and

\[
|y| \cdot |z - \frac{y}{|y|^2} (y; z)| = |y| \cdot |z| \cdot |\sin(y, z)|
\]

\[
= |\sin(u, v) \cdot \sin(v, w) \cdot \sin \angle uvw|.
\]

The second part of (7) follows from \(|A \cdot \sin \alpha| + |B \cdot \cos \alpha| \leq (A^2 + B^2)^{1/2}\). The sin-factors in (8) are trivial since with \(u' = (u - v(u; v)) \cdot |\sin(u, v)|^{-1}, x' = (x - w(x; w)) \cdot |\sin(x, w)|^{-1}\) we have

\[
(R(u, v)x; w) = |\sin(u, v) \cdot \sin(x, w)| \cdot (R(u', v)x'; w).
\]
We apply (7) to the four terms on the right of (6) (with \( u', x' \) instead of \( u, x \)) and since \( w \perp u', v, x' \) all the \( K \cdot \cos \)-terms vanish. Hence (8) follows using \((v+x')^2+(v-x')^2+(u'+x')^2+(u'-x')^2=8\). In (9) we can assume \( u \perp v, x \perp w \) (replace \( u, x \) by \( u', x' \)). Put \( y = u - x(u; x), \ z = v - x(v; x), \ a = u(v; x) - v(u; x) \). Note \( a \perp x \) and \( y(v; x) - v(u; x) = a - x(u; x)(v; x) \). Then

\[
(R(u, v)x; w) = (R(y, z)x; w) + (R(y, x(v; x))x; w) + (R(x(u; x), v)x; w)
= (R(y, z)x; w) + (R(a, x)x; w).
\]

From (8) we have

\[
| (R(y, z)x; w) | \leq \frac{3}{5} (\Delta - \delta) | y | | z | \sin(y, z)
= \frac{3}{5} (\Delta - \delta) | \sin(u, x) \cdot \sin(v, x) \cdot \sin \angle uxv | ,
\]
and from (7) we have

\[
| (R(a, x)x; w) | \leq | a | \cdot (K^2 + \frac{1}{4} (\Delta - \delta)^2)^{1/2}.
\]

Observe \( |a|^2 = (u; x)^2 + (v; x)^2 \leq |x|^2 = 1 \) and—from the spherical cosine formula for the triangle \( u, x, v \)—

\[
| \sin(u, x) \cdot \sin(v, x) \cdot \sin \angle uxv |^2 = 1 - | a |^2.
\]

Then (9) follows again from \( |A \cdot \sin \alpha | + | B \cdot \cos \alpha | \leq (A^2 + B^2)^{1/2} \).

**References**


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