ON THE CONVEXITY OF LEMNISCATES

DOROTHY BROWNE SHAFFER

Abstract. Let $L_i$ denote the lemniscate $|\prod_{i=1}^{n}(z-\xi_i)| = L$. Assume the poles $\xi_i$ are inscribed in the disc $|z| \leq a$. Let $z_0 = n^{-1}\sum_{i=1}^{n}\xi_i$. Conditions for the convexity of $L_1$ are established in terms of $a$ and $z_0$. Sharp bounds are derived for real $\xi_i$.

Let $L_1$ be the lemniscate $L_1: |p(z)| = |\prod_{i=1}^{n}(z-\xi_i)| = 1$. It was proved by Erdös, Herzog and Piranian \cite{1} that $L_1$ is convex if all the $\xi_i$ are inscribed in a disc of radius $a \leq \sin \pi/8/(1+\sin \pi/8)$. This estimate was improved by the author \cite{3} to $a \leq 2^{1/2} - 1 = .414$. It is the object of this note to improve these bounds; a sharp result is obtained for the case of a real polynomial.

**Theorem.** $L_1$ is convex if $2^{1/2} - 1 \leq a \leq 1/3^{1/2}$ and

$$|z_0| \leq (1 - 3a^2)/(23^{1/2}a)$$

where $z_0 = n^{-1}\sum_{i=1}^{n}\xi_i$.

Proof. The author proved \cite{3} that any lemniscate, with its zeros inscribed in a disc of radius $a$ is convex if it lies outside of a concentric circle of radius $2^{1/2} a$. By a lemma due to Pommerenke \cite{2}, $L_1$ contains the disc $|z-z_0| \leq (1-a^2+|z_0|^2)^{1/2}$, if $a^2-|z_0|^2 \leq 1$.

It follows that $L_1$ lies outside the disc with center at the origin, radius

$$1 - a^2 + |z_0|^{1/2} - |z_0|$$

and $L_1$ is convex if

$$|z_0|^2)^{1/2} - |z_0| \geq 2^{1/2}a.$$
Theorem 2. Assume in addition all the $\xi_i$ real, then $L_1$ is convex if either $a \leq 1/2$ or $1/2 \leq a \leq 1/2^{1/2}$ with $|z_0|^{(1-2a^2)/2a}$.

The proof follows from the fact that the author's proof of the convexity condition implies that a lemniscate is convex at $z$ if the angle subtended at $z$ by any pair of zeros is acute. It follows that for $L_1$ with all the zeros on a diameter of the disc, $L_1$ is convex if it lies outside the same disc. Applying condition (2) this will be true if $(1-a^2+|z_0|^2)^{1/2}-|z_0| \geq a$, and the statement of the theorem follows by solution of the inequality.

These bounds are sharp, they are approached for large $m$ by the lemniscate $|p(z)| = |(z-\frac{1}{2})^m(z+\frac{1}{2})| = 1$ and for $z_0=0$ by $|(z-2^{-1/2})(z+2^{-1/2})| = 1$.

References


Courant Institute of Mathematical Sciences, New York University, New York, New York 10012