ON THE CONVEXITY OF LEMNISCATES

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Abstract. Let $L_1$ denote the lemniscate $|\prod_{i=1}^{n}(z-\xi_i)| = 1$. Assume the poles $\xi_i$ are inscribed in the disc $|z| \leq a$. Let $z_0 = n^{-1}\sum_{i=1}^{n} \xi_i$. Conditions for the convexity of $L_1$ are established in terms of $a$ and $z_0$. Sharp bounds are derived for real $\xi_i$.

Let $L_1$ be the lemniscate $L_1: |p(z)| = |\prod_{i=1}^{n}(z-\xi_i)| = 1$. It was proved by Erdős, Herzog and Piranian [1] that $L_1$ is convex if all the $\xi_i$ are inscribed in a disc of radius $a \leq \sin \pi/8/(1+\sin \pi/8)$. This estimate was improved by the author [3] to $a \leq 2^{1/2} - 1 = .414$. It is the object of this note to improve these bounds; a sharp result is obtained for the case of a real polynomial.

Theorem 1. $L_1$ is convex if $2^{1/2} - 1 \leq a \leq 1/3^{1/2}$ and

$$|z_0| \leq (1 - 3a^2)/(2^{3/2}a) \quad \text{where} \quad z_0 = n^{-1}\sum_{i=1}^{n} \xi_i.$$

Proof. The author proved [3] that any lemniscate, with its zeros inscribed in a disc of radius $a$ is convex if it lies outside of a concentric circle of radius $2^{1/2}a$. By a lemma due to Pommerenke [2], $L_1$ contains the disc $|z-z_0| \leq (1-a^2+|z_0|^2)^{1/2}$, if $a^2-|z_0|^2 \leq 1$.

It follows that $L_1$ lies outside the disc with center at the origin, radius

$$(1 - a^2 + |z_0|^2)^{1/2} - |z_0|$$

and $L_1$ is convex if

$$(1 - a^2 + |z_0|^2)^{1/2} - |z_0| \geq 2^{1/2}a.$$ 

Inequality (3) solved for $|z_0|$ gives condition (1).

If $|z_0| = 0$, i.e. the center of gravity of the zeros is assumed to be the center of the disc containing the zeros we obtain that $L_1$ is convex if $a \leq 1/3^{1/2}$. If $z_0$ is allowed to approach the boundary, $|z_0| = a$, the previous condition $a \leq 2^{1/2} - 1$ follows.

Presented to the Society, August 28, 1969 under the title Geometric properties of equipotential surfaces and curves; received by the editors November 10, 1970.

AMS 1969 subject classifications. Primary 3010.

Key words and phrases. Lemniscate, level line, convexity.

1 During the course of this work the author was a National Science Foundation Faculty Fellow. Publication supported by NSF grant GP-23504, Fairfield University.
Theorem 2. Assume in addition all the \( \xi_i \) real, then \( L_1 \) is convex if either \( a \leq 1/2 \) or \( 1/2 \leq a \leq 1/2^{1/2} \) with \( |z_0| \leq (1 - 2a^2)/2a. \)

The proof follows from the fact that the author's proof of the convexity condition implies that a lemniscate is convex at \( z \) if the angle subtended at \( z \) by any pair of zeros is acute. It follows that for \( L_1 \) with all the zeros on a diameter of the disc, \( L_1 \) is convex if it lies outside the same disc. Applying condition (2) this will be true if \( (1 - a^2 + |z_0|^2)^{1/2} - |z_0| \geq a \), and the statement of the theorem follows by solution of the inequality.

These bounds are sharp, they are approached for large \( m \) by the lemniscate \( |p(z)| = |(z - \frac{1}{2})^m(z + \frac{1}{2})| = 1 \) and for \( z_0 = 0 \) by \( |(z - 2^{-1/2})(z + 2^{-1/2})| = 1. \)

References


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