Abstract. If a space $X$ is an $H$-space, then the homotopy suspension homomorphism is a monomorphism onto a direct factor in all dimensions. We present an example to show that the converse is false.

1. Introduction. Let $X$ be an $H$-space and a connected CW-complex. Then the inclusion map $i: X \to \Omega \Sigma X$, defined by $i(x)(t) = (x, t) \in \Sigma X$, has a left homotopy inverse. (This was first proved for countable CW-complexes by I. M. James in [6]. For a semisimplicial argument which removes the hypothesis of countability see [4, p. 208] or, for a geometric argument, see [3].) It follows that the suspension homomorphism $i_*: \pi_m(X) \to \pi_m(\Omega \Sigma X)$ is a monomorphism onto a direct factor for all $m$. We call a space $Y$ which satisfies this algebraic criterion ($i_*$ is a monomorphism onto a direct factor in all dimensions) a $\Sigma$-space and we call $Y$ a $\sigma$-space if $i_*$ is a monomorphism in all dimensions. The purpose of this note is to exhibit a $\Sigma$-space which is not an $H$-space.

As one would expect, the example we have in mind is an infinite CW-complex and we ask whether or not there are finite CW-complexes which are $\Sigma$-spaces and not $H$-spaces. This question is similar to one raised by G. T. Porter: A $\Sigma$-space (in fact a $\sigma$-space) has the property that its higher order spherical Whitehead products (HOWP) vanish [8, Corollary 4]. In [9], Porter asked whether there are finite CW-complexes having trivial HOWP which are not $H$-spaces; to our knowledge, Porter's question is still unresolved.

In what follows we will denote inclusion maps by $i, i'$ etc. and Hurewicz homomorphisms by $h, h'$ etc.

2. The example. Consider the two-stage Postnikov system

$$K(Z_2, 2n - 1) \to E_n \to K(Z, 2) \quad (n \geq 2)$$

with $k$-invariant $\alpha^n$, where $\alpha$ is the generator of $H^2(Z, 2; Z_2)$.
**Theorem 1.**

1. \( E_n \) is a \( \Sigma \)-space.
2. \( E_n \) is a \( \sigma \)-space for all \( n \).
3. \( E_n \) is an \( H \)-space if and only if \( n = 2^k \).

Part (3) of Theorem 1 follows from the fact that the \( k \)-invariant \( \alpha^n \) is primitive if and only if \( n = 2^k \) \cite{7}. As a first step to proving parts (1) and (2) of Theorem 1, we replace \( E_n \) by an equivalent \( \text{CW} \)-complex.

Let \( CP^{n-1} \) denote the \((n-1)\)-dimensional complex projective space and \( p_{n-1} : S^{2n-1} \to CP^{n-1} \) the usual fibration with fiber \( S^1 \). Then \( C \Sigma n = CP^{n-1} \cup_{p_{n-1}} e^{2n} \). Starting with \( CP^1 = S^2 \) and the Hopf map \( \phi^1 : S^3 \to S^2 \) we see that \( CP^n \) has a \( \text{CW} \)-structure with exactly one cell in each even dimension \( \leq 2n \). Let \( d_{n-1} : S^{2n-1} \to S^{2n-1} \) be a map of degree 2 and set \( X_n = CP^{n-1} \cup_{p_{n-1}, d_{n-1}} e^{2n} \). Let \( Y_n \) be the space obtained from \( X_n \) by attaching \( m \)-cells, \( m \geq 2n + 1 \), so as to "kill" its homotopy groups in dimensions \( \geq 2n \).

**Lemma 1.** The spaces \( Y_n \) and \( E_n \) have the same homotopy type.

**Proof.** Since \( p_{n-1} d_{n-1} \) represents \( \pm 2 \in Z = \pi_{2n-1}(CP^{n-1}) \), it is clear that \( Y_n \) and \( E_n \) have the same homotopy groups and so \( Y_n \) also has a Postnikov system of the form

\[
K(Z, 2n - 1) \to E_n' \to K(Z, 2).
\]

Furthermore, there is a map \( Y_n \to E_n' \) which is a homotopy equivalence. Since \( H^{2n}(Z, 2; Z_2) = Z_2 \), the \( k \)-invariant of this Postnikov system is either 0 or \( \alpha^n \). If the \( k \)-invariant were 0, we would have \( E_n = K(Z, 2) \times K(Z, 2n - 1) \), but then \( H^{2n-1}(E_n; Z_2) = Z_2 \) whereas \( H^{2n-1}(Y_n; Z_2) = 0 \), since \( Y_n \) has no \((2n-1)\)-cells. We conclude that the \( k \)-invariant is \( \alpha^n \) and that \( E_n' = E_n \) which establishes the lemma.

**Lemma 2.** \( \iota_* : \pi_2(Y_n) \to \pi_2(\Omega \Sigma Y_n) \) is an isomorphism.

**Proof.** This follows from the homotopy suspension theorem.

It remains to consider \( \iota_* : \pi_{2n-1}(Y_n) \to \pi_{2n-1}(\Omega \Sigma Y_n) \) or equivalently, by a cellular approximation argument, \( \iota_* : \pi_{2n-1}(X_n) \to \pi_{2n-1}(\Omega \Sigma X_n) \).

From the definition of \( X_n \) and \( CP^n \) we see that there is a map \( f_n : X_n \to CP^n \) which maps the subspace \( CP^{n-1} \) of \( X_n \) and \( CP^n \) identically. In the following we will consider \( f_n \) to be an inclusion and, by abuse of notation, will consider the "pair" \( (CP^n, X_n) \). Set \( A_n = \Omega \Sigma X_n, B_n = \Omega \Sigma CP^n \) and \( g_n = \Omega \Sigma f_n : A_n \to B_n \).

**Lemma 3.** In the following diagram all the homomorphisms are isomorphisms and the groups are isomorphic to \( Z_2 \).

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Theorem. Clearly

\[ H_m(X_n) = H_m(CP^n) = \mathbb{Z} \quad m = 0, 2, \ldots, 2n, \]

\[ = 0 \quad \text{otherwise.} \]

Moreover, we may choose generators \( a_{2m} \) and \( b_{2m} \) of \( H_{2m}(X_n) \) and \( H_{2m}(CP^n) \), respectively, so that

\[ f_n^*(a_{2m}) = b_{2m} \quad 0 \leq m \leq n - 1, \]

\[ = 2 \cdot b_{2n} \quad m = n. \]

Consequently, the pair \((CP^n, X_n)\) is \((2n-1)\)-connected; by the Hurewicz theorem \( h_1 \) is an isomorphism. Plainly \( H_{2n}(CP^n, X_n) = \mathbb{Z}_2 \).

The Pontrjagin ring \( H_*(A_n) \) (\( H_*(B_n) \)) is the free associative algebra on \( n \) generators, namely \( \alpha_2, \ldots, \alpha_{2n} \) (resp. \( \beta_2, \ldots, \beta_{2n} \)), where \( \alpha_{2m} = i_*(a_{2m}) \) and \( \beta_{2m} = i_*(b_{2m}) \), \( m = 1, \ldots, n \) [1]. Since the map \( g_n \) is an \( H \)-map, it follows that the induced homomorphism \( g_n^* : H_*(A_n) \rightarrow H_*(B_n) \) is an isomorphism for \( m \leq 2n - 1 \) and has cokernel \( \mathbb{Z}_2 \) in dimension \( 2n \). As above we conclude that \( h_2 \) is an isomorphism and that \( H_{2n}(B_n, A_n) = \mathbb{Z}_2 \). Finally, the homomorphism \( i_* \) is easily seen to be an epimorphism and therefore an isomorphism. This completes the proof of the lemma.

Consider Figure 1. That the leftmost vertical homomorphism

\[ \pi_{2n}(A_n) \rightarrow H_{2n}(A_n) \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \pi_{2n}(B_n) \rightarrow H_{2n}(B_n) \]

\[ \pi_{2n}(CP^n, X_n) \rightarrow \pi_{2n}(B_n, A_n) \rightarrow H_{2n}(B_n, A_n) \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \pi_{2n-1}(X_n) \rightarrow \pi_{2n-1}(A_n) \rightarrow H_{2n-1}(A_n) \]

\[ \downarrow \]

\[ \pi_{2n-1}(B_n) \]

Figure 1

is an isomorphism follows from the fact that \( \pi_{2n}(CP^n) = \pi_{2n-1}(CP^n) = 0 \) and by Lemma 3 the homomorphisms in the third row are isomor-
phisms. It follows that \( \iota_\bullet \) is a monomorphism if and only if \( j_\bullet h = 0 \) and that \( \iota_\bullet \) is an epimorphism if and only if \( \pi_{2n-1}(B_n) = \pi_{2n}(\Sigma CP^n) = 0 \).

**Proof of part (2) of Theorem 1.** By the foregoing remark it suffices to show that \( j_\bullet h = 0 \). Now any element of \( H_{2n}(B_n) \) can be expressed as \( r \cdot \beta_{2n} + g_{n*}(y) \), \( r = 0, 1 \), \( y \in H_{2n}(A_n) \) (see the proof of Lemma 3). To show \( j_\bullet h = 0 \), it suffices to show that no element of the form \( \beta_{2n} + g_{n*}(y) \) is in the image of \( h \). Let \( x \in \pi_{2n}(B_n) \) and, in order to obtain a contradiction, assume that \( h(x) = \beta_{2n} + g_{n*}(y) \). Write \( y = \alpha_{2n} + z \), where \( z \) is in the subalgebra of \( H_*(A_n) \) generated by the set \( \{ \alpha_2, \ldots, \alpha_{2n-2} \} \) and \( s \) is some integer. Consider Figure 2,

\[
\begin{array}{ccc}
\pi_{2n}(B_n) & \xrightarrow{\sigma'} & \pi_{2n+1}(\Sigma CP^n) \\
\downarrow h & & \downarrow h' \\
H_{2n}(B_n) & \xrightarrow{\sigma} & H_{2n+1}(\Sigma CP^n) \\
\iota_\bullet & & h'' \\
\iota_\bullet & & \iota_\bullet \\
H_{2n}(CP^n) & & H_{2n+1}(CP^n)
\end{array}
\]

**Figure 2**

where \( \sigma \) denotes the homology suspension, \( \sigma' \) denotes the usual bijection and \( \iota' \) denotes the inclusion map. (The special unitary group \( SU(n+1) \) can be given a CW-structure with \( \Sigma CP^n \) as a subcomplex.) From the (signed) commutativity of Figure 2 and the relations \( g_{n*}(\alpha_{2n}) = 2 \cdot \beta_{2n} \) and \( \sigma(g_{n*}(z)) = 0 \) [11, Corollary 6.2] we see that \( h'' \iota' \sigma'(x) = (2s + 1) \cdot \iota' \Sigma(b_{2n}) \). Toda [10] has shown that there is an element \( \xi \in \pi_{2n+1}(\Sigma CP^n) \) such that \( \iota_\bullet \xi(\xi) \) generates \( \pi_{2n+1}(SU(n+1)) = Z \) and \( h'(\xi) = n! \cdot \Sigma(b_{2n}) \). Therefore, there is an integer \( t \) such that \( \iota_\bullet \sigma'(x) = t \cdot \iota_\bullet \xi(\xi) \) and we compute \( h'' \iota_\bullet \sigma'(x) = t \cdot n! \cdot \iota_\bullet \Sigma(b_{2n}) \). Since \( \iota_\bullet \Sigma(b_{2n}) \) is of infinite order in \( H_{2n+1}(SU(n+1)) \) we cannot have an even multiple of this element equal to an odd multiple. This gives the desired contradiction and completes the proof of part (2) of Theorem 1.

**Lemma 4.** \( \pi_{2n}(\Sigma CP^n) = 0 \) if and only if \( \pi_{2n}(\Sigma CP^{n-1}) = Z_{n!} \).

**Proof.** Consider the following exact sequence due to J. H. C. Whitehead [5, p. 115]

\[
\pi_{2n}(\Sigma CP^n) \rightarrow \pi_{2n+1}(\Sigma CP^n) \rightarrow H_{2n+1}(\Sigma CP^n) \rightarrow \Gamma_{2n}(\Sigma CP^n) \rightarrow \pi_{2n}(\Sigma CP^n) \rightarrow H_{2n}(\Sigma CP^n) \rightarrow.
\]

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It follows from the definition of $\Gamma_{2n}$ that $\Gamma_{2n} = \pi_{2n}(\Sigma CP^{n-1})$. Since $H_{2n}(\Sigma CP^n) = 0$, the lemma will follow if we show that $\text{Im}(h') = n! \cdot Z \subset Z = H_{2n+1}(\Sigma CP^n)$. But this follows from the result of Toda cited above and the fact that $h''$ (see Figure 2) is a monomorphism [2].

Proof of part (1) of Theorem 1. As noted above, it suffices to show that $\pi_6(\Sigma CP^3) = 0$ or, by Lemma 4, that $\pi_6(\Sigma CP^2) = Z_6$. One can compute $\pi_6(\Sigma CP^3) = Z_6$ by considering the homotopy sequence of the pair $(\Sigma CP^3, \Sigma S^3)$ and making use of the fact that $Z_6$ is a subgroup of $\pi_6(\Sigma CP^1)$ (see the proof of Lemma 4).

In conclusion we point out a contrast between the Hurewicz homomorphism and the suspension homomorphism. We make no attempt to be precise here; precise statements are given in [3].

Let $X^\infty$ denote the infinite symmetric product space of $X$ and let $\iota: X \to X^\infty$ be the inclusion map. As is well known, the geometric condition that $X$ be dominated by $X^\infty$ is equivalent to the associated algebraic condition that the induced homomorphism $\iota_*: \pi_m(X) \to \pi_m(X^\infty) \cong H_m(X)$ (i.e. the Hurewicz homomorphism) be a monomorphism onto a direct factor in all dimensions.

Now let $X_\infty$ denote the reduced product space of $X$ and let $\iota: X \to X_\infty$ be the inclusion map. Our example shows that the geometric condition that $X$ be dominated by $X_\infty$ (i.e. that $X$ be an $H$-space) is not equivalent to the associated algebraic condition that the induced homomorphism $\iota_*: \pi_m(X) \to \pi_m(X_\infty) \cong \pi_{m+1}(\Sigma X)$ (i.e. the suspension homomorphism) be a monomorphism onto a direct factor in all dimensions.

Bibliography


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