COMPLETENESS OF HAMILTONIAN VECTOR FIELDS

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Abstract. We prove that under certain conditions the flow of a Hamiltonian vector field on a possibly infinite-dimensional dynamical system exists for all time.

We shall prove that under certain natural conditions the Hamiltonian vector field of a (possibly infinite-dimensional) dynamical system has integral curves which extend for all time. W. Gordon [3] has proved the result in the case of Euclidean space. We are grateful to J. Marsden for suggesting the problem and to W. Meyer for useful conversations.

Let $M$ be a smooth Hilbert manifold with a Riemannian structure, $\langle \cdot, \cdot \rangle$, let $V_0 : M \rightarrow \mathbb{R}$ be a smooth function and let $V = V_0 \circ \pi$ where $\pi : T(M) \rightarrow M$ is the projection map of the tangent bundle. Define $H : T(M) \rightarrow \mathbb{R}$ by $H = K + V$ where $K$, the kinetic energy, is defined by $K(V) = \frac{1}{2} \langle V, V \rangle$.

It is well known that $T^*M$ (and by means of $\langle \cdot, \cdot \rangle$, $TM$) has a natural symplectic two-form $\Omega$, and the equation $i(Z)\Omega = dH$ can be used to define a vector field $Z$ on $TM$ [4, p. 110].

**Theorem.** If $M$ is complete under the metric induced by $\langle \cdot, \cdot \rangle$ and if $V$ is bounded below, then all integral curves of $Z$ extend for all time. Equivalently the flow $F : TM \times \mathbb{R} \rightarrow TM$ which satisfies $(dF/dt)(X, s) = Z_{F(X, s)}$ is defined on all of $TM \times \mathbb{R}$.

To prove the theorem we must construct a Riemannian structure for $TM$ and this we do by use of the affine connection.

It is well known that $\langle \cdot, \cdot \rangle$ defines a unique connection and for each $p \in M$ and $X \in T_pM$, $V$ defines a splitting of $T_xT_pM$ into vertical and horizontal subspaces; i.e. $T_xT_pM = V \oplus \mathcal{H}$, [2]. Also there are canonical linear isomorphisms $L : V \rightarrow T_pM$ and $T\pi : \mathcal{H} \rightarrow T_pM$, $T\pi$ being the tangent map to $\pi$. We define an inner product, $\langle \cdot, \cdot \rangle$, on $T_xT_pM$ by declaring $V$ and $\mathcal{H}$ perpendicular and $L$ and $T\pi$ isometries. This clearly defines a Riemannian structure for $T(M)$.

**Lemma.** $\langle \cdot, \cdot \rangle$ induces a complete metric on $TM$.

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Proof. Let \( \{X_i\} \) be a Cauchy sequence in \( TM \) and let \( p_i = \pi(X_i) \). For any \( q \in M \) and \( y \in T_q M \), \( T\pi : T_q TM \to T_q M \) is distance decreasing. Hence \( \pi \) decreases the length of curves, so \( \{p_i\} \) is Cauchy in \( M \).

Let \( p = \lim_{i \to \infty} \{p_i\} \) and choose a closed neighborhood \( U \) about \( p \) and a chart \( \varphi : U \to E \). (\( E \) is a Hilbert space on which \( M \) is modeled.) \( \varphi \) induces a chart \( T\varphi : T(U) \to E \times E \) in a standard way [4, p. 40], and for \( i \) sufficiently large \( X_i \in T(U) \). We can assume that \( \varphi(U) \) is a closed ball in \( E \) in which case \( T\varphi(T(U)) = \varphi(U) \times E \) is a closed subset of \( E \times E \). \( \varphi(U) \times E \) is therefore complete as a metric subspace of \( E \times E \), so if we can show \( \{T\varphi(X_i)\} \) is Cauchy in \( \varphi(U) \times E \) then \( \{T\varphi(X_i)\} \) will converge and so will \( \{X_i\} \).

To show \( \{T\varphi(X_i)\} \) Cauchy, we make \( \varphi : U \to E \) a normal coordinate chart so that \( TT\varphi: T_q TM \to U \times E \times E \times E \) is an isometry for all \( y \in \pi^{-1}(p) \), and so that there exists a constant \( A \) such that for any \( W \in TTU \), \( \|T\varphi W\| \leq A\|W\| \). The existence of such an \( A \) means that all curves in \( TU \) are stretched, under the map \( T\varphi \), by at most a factor of \( A \). Then, letting \( \rho \) be the metric on \( TM \), we must show that for \( i, j \) sufficiently large and for any curve \( \gamma \) of length sufficiently close to \( \rho(X_i, X_j) \), \( \gamma \) is contained in \( TU \). If this is true, \( T\varphi(\gamma) \) has length at most \( A \rho(X_i, X_j) \) and \( T\varphi(X_i) \) is at most \( A \rho(X_i, X_j) \).

To show curves such as \( \gamma \) are contained in \( T(U) \) we assume \( U \) is a metric ball about \( p \) of radius \( 3\varepsilon \). Then let \( V \subseteq U \) be the ball of radius \( \varepsilon \), and pick \( i, j \) sufficiently large so that \( \rho(X_i, X_j) < \varepsilon \) and \( X_i, X_j \in T(V) \). Then if \( \gamma \) is a curve from \( X_i \) to \( X_j \) of length less than \( 2\varepsilon \), \( \gamma_0 = \pi \circ \gamma \) must also have length less than \( 2\varepsilon \) so \( \gamma_0 \subseteq U \). Thus \( \gamma \subseteq TU \). \( T\varphi(\gamma) \) has length at most \( 2A\varepsilon \), and the distance from \( T\varphi(X_i) \) to \( T\varphi(X_j) \) is therefore at most \( 2\varepsilon \). Hence \( \{T\varphi(X_i)\} \) is Cauchy and the lemma is proven.

Proof of Theorem. Consider the equation \( i(Z)\Omega = dH \) and let \( Z = S + G \) where \( S \) satisfies \( i(S)\Omega = dK \) and \( G \) satisfies \( i(G)\Omega = dV \). Then \( S \) is the spray of the Riemannian metric \( \langle , \rangle \) [4, p. 110], and \( G \) is the vertical lift of grad \( V \); i.e. \( G_z = L^{-1}(\text{grad } V) \) where \( L : U \to T_{\pi(z)}M \). (See [1].)

Let \( c : (a, b) \to TM \) be a maximal integral curve of \( Z \). We wish to show that \( a = -\infty, b = \infty \), and to do this, we first assume \( a \) (respectively \( b \)) finite and show \( \lim_{t \to a} \{c(t)\} \) (respectively \( \lim_{t \to b} c(t) \)) exists in \( TM \). Then by the fundamental theorem of ordinary differential equations [4], the domain of \( c \) extends to \( (a + \varepsilon, b) \) (respectively \( (a, b + \varepsilon) \)). This contradicts the maximality of \( (a, b) \) and proves the theorem.

The main fact needed to show \( \lim_{t \to a} \{c(t)\} \) exists, is that \( H \) is constant along \( c \). This is simply conservation of energy and is proven.
in [1, p. 110]. Let $N$ be the lower bound for $V$ and let $N_1 = H(c)$. Then $K \uparrow c$ is bounded by $N_1 - N$.

Let $c_0 = \pi \circ c$, a curve in $M$. Then $c_0'$, the tangent vector to $c_0$ satisfies $c_0'(t) = T\pi(c'(t)) = T\pi(Z_{c(t)})$. But $T\pi(Z_{c(t)}) = T\pi(S_{c(t)}) = c(t)$ since $G$ is vertical and $S$ is a spray. Therefore since $K$ is bounded on $c$, the length of $c_0'(t)$ is bounded and, since $M$ is complete, $\lim_{t \to b} \{c_0(t)\}$ exists. Let $c_0(b) = \lim_{t \to b} \{c_0(t)\}$.

Now we seek a limit of $c(t)$ in $\pi^{-1}(c_0(b))$. To show that it exists we must only show that $c'(t)$ is bounded in the metric on $TM$. $c'(t) = Z_{c(t)} = S_{c(t)} + G_{c(t)}$, and $S$ and $G$ are horizontal and vertical respectively. Also $G_{c(t)}$ depends only on $c_0(t)$ and since $c_0(t)$ extends continuously to $c_0(b)$, $G_{c(t)}$ remains bounded as $t \to b$. Also $\|S_{c(t)}\| = \|c(t)\|$, from the definition of the metric on $TM$. It follows that $\|c'(t)\| = \|S_{c(t)}\| + \|G_{c(t)}\|$ remains bounded as $t \to b$, so $c(t)$ converges. The same argument works as $t \to a$, and the theorem follows.

**Remark 1.** If $M$ is finite-dimensional, one can use local compactness of $TM$ instead of a metric on $TM$, so the proof is much shorter.

**Remark 2.** It is easy to show that the theorem is false if $M$ is not complete or $V$ is not bounded below. There are counterexamples even if $M$ is one-dimensional.

**References**


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