UNIQUENESS OF GENERATORS OF PRINCIPAL IDEALS
IN RINGS OF CONTINUOUS FUNCTIONS

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Abstract. Let \( aR \) denote the principal right ideal generated in a ring \( R \) by an element \( a \). Kaplansky has raised the question: If \( aR = bR \), are \( a \) and \( b \) necessarily right associates? In this note we show that for rings of continuous functions the answer is affirmative if and only if the underlying topological space is zero-dimensional. This gives an algebraic characterization of the topological concept "zero-dimensional". By extending the notion of uniqueness of generators of principal ideals we are able to give an algebraic characterization of the concept "\( n \)-dimensional".

In the sequel, \( R \) is assumed to be commutative.

Definition. A set of principal ideals \( \{a_iR\}, i = 1, \ldots, n \), is uniquely generated if whenever \( a_iR = b_iR \), \( i = 1, \ldots, n \), there exist elements \( u_i \) of \( R \) such that \( a_i = b_iu_i \), \( i = 1, \ldots, n \), and \( u_1R + \cdots + u_nR = R \). The dimension of \( R \)—denoted by \( \dim R \)—is the least integer \( n \) such that every set of \( n+1 \) principal ideals is uniquely generated.

In the following \( X \) will denote a completely regular topological space, \( C(X) \) the ring of real-valued continuous functions defined on \( X \), and \( C^*(X) \) the subring of \( C(X) \) consisting of the bounded functions in \( C(X) \). For \( f \in C(X) \), the zero set \( Z(f) \) of \( f \), is defined by \( Z(f) = \{ x \in X : f(x) = 0 \} \). Clearly \( f_1C(X) + \cdots + f_nC(X) = C(X) \) if and only if \( Z(f_1) \cap \cdots \cap Z(f_n) = \emptyset \). For further information on zero sets the reader is referred to [2]. An important fact is that disjoint zero sets are completely separated.

We use the modification of covering dimension involving basic covers [2, p. 243], and the equivalent definitions given in [1]. The unit cube in \( E^{n+1} \) is denoted by

\[ I_{n+1} : I_{n+1} = \{ x \in E^{n+1} : -1 \leq x_i \leq 1, i = 1, \ldots, n + 1 \} \]

We also write

\[ I_+^{n+1} = \{ x \in E^{n+1} : 0 \leq x_i \leq 1, i = 1, \ldots, n + 1 \} \]

and

\[ S_n = \{ x \in I_+^{n+1} : x_i = 0 \text{ or } 1 \text{ for some } i \} \].

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Theorem. The following conditions are equivalent.

(i) \( \dim X = n \).

(ii) \( \dim C(X) = n \).

(iii) \( \dim C^*(X) = n \).

Proof. \( \dim C(X) \leq \dim X \). Let \( \dim X = n \) and suppose \( f_i C(X) = g_i C(X) \), \( i = 1, \ldots, n + 1 \). There exist functions \( s_i, t_i \) in \( C(X) \) such that \( f_i = g_i s_i, g_i = f_i t_i, i = 1, \ldots, n + 1 \). Then \( g_i = g_i s_i t_i \), so that \( g_i (1 - s_i t_i) = 0 \). Since \( Z_i = Z(s_i) \) and \( Z'_i = Z(s_i + 1 - s_i t_i) \) are disjoint zero sets, there exist \( m_i \) in \( C(X) \) such that \( m_i \) is 1 on \( Z_i \), 0 on \( Z'_i \) and \( 0 \leq m_i \leq 1 \). Let \( m \) be the mapping of \( X \) into \( Z^{n+1}_+ \) defined by \( m(x) = (m_1(x), \ldots, m_{n+1}(x)) \). Then \( m \) maps all the points in \( Z_i \) and \( Z'_i \) into \( S^*_+ \). Since \( \dim X \leq n \), there exists a mapping \( h = (h_1, \ldots, h_{n+1}) \) of \( X \) into \( S^*_+ \) such that \( h(x) = m(x) \) whenever \( m(x) \in S^*_+ \) [1, Definition 3]. Let \( k_i = s_i + h_i (1 - s_i t_i), i = 1, \ldots, n + 1 \). Then \( Z(k_i) \cap \cdots \cap Z(k_{n+1}) = \emptyset \). To see this, note that for any \( x \in X \) we have \( h_i(x) = 0 \) or \( h_i(x) = 1 \) for some \( i \). If \( h_i(x) = 0 \) then \( s_i(x) \neq 0 \) so that \( k_i(x) \neq 0 \). If \( h_i(x) = 1 \), then \( s_i(x) + 1 - s_i(x) t_i(x) \neq 0 \) and again \( k_i(x) \neq 0 \). Hence \( k_i C(X) + \cdots + k_{n+1} C(X) = C(X) \). Finally, \( g_i k_i = g_i s_i + h_i (1 - s_i t_i) = g_i s_i = f_i, i = 1, \ldots, n + 1 \).

\( \dim X \leq \dim C(X) \). Let \( \dim C(X) = n \) and let \( Z_i, Z'_i, i = 1, \ldots, n + 1 \), be disjoint pairs of zero sets of \( X \). We construct functions \( f_i, k_i \) such that \( f_i = k_i \left| f_i \right|, f_i = 1 \) on \( Z_i, f_i = -1 \) on \( Z'_i, i = 1, \ldots, n + 1 \). There exist \( p_i \) such that \( p_i = 1 \) on \( Z_i \) and \( p_i = -1 \) on \( Z'_i \). There exist \( k_i, s_i, t_i \) such that \( k_i = 1 \) when \( p_i \geq \frac{1}{2}, k_i = -1 \) when \( p_i \leq -\frac{1}{2}, 1 \leq k_i \leq 1: \)

\[ t_i = 1 \quad \text{on} \quad Z_i, t_i = 0 \quad \text{when} \quad p_i \leq \frac{1}{2}, 0 \leq t_i \leq 1, \]

\[ s_i = -1 \quad \text{on} \quad Z'_i, s_i = 0 \quad \text{when} \quad p_i \leq -\frac{1}{2}, -1 \leq s_i \leq 0. \]

Let \( f_i = s_i + t_i \). Then \( f_i = k_i \left| f_i \right|, \left| f_i \right| = k_i f_i \), so that \( f_i C(X) = \left| f_i \right| C(X), i = 1, \ldots, n + 1 \). Since \( \dim C(X) \leq n \), there exist \( h_1, \ldots, h_{n+1} \) such that \( f_i = h_i \left| f_i \right|, i = 1, \ldots, n + 1 \), and \( h_1 C(X) + \cdots + h_{n+1} C(X) = C(X) \). Clearly \( Z_i \) and \( Z'_i \) are separated in \( X - Z(h_i) \) and \( Z(h_i) \cap \cdots \cap Z(h_{n+1}) = \emptyset \). Hence \( \dim X \leq n \) [1, Definition 4].

\( \dim X = \dim C^*(X) \). This is proved using the same methods as above, with \( C(X) \) replaced by \( C^*(X) \).

A simple consequence of this theorem is the following well-known result.

Corollary. \( \dim X = \dim \beta X = \dim vX \), where \( \beta X \) is the Stone-Čech compactification of \( X \), and \( vX \) is the Hewitt real-compactification of \( X \).

Proof. Since \( C^*(X) \) is isomorphic to \( C(\beta X) \), we have \( \dim X \)
= \dim C^*(X) = n \iff \dim C(\beta X) = n \iff \dim \beta X = n. Since C(X) is isomorphic to C(uX), we have \dim X = n \iff \dim C(X) = n \iff \dim C(uX) = n \iff \dim uX = n.

REFERENCES


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