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CAN AN LCA GROUP BE ANTI-SELF-DUAL?

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ABSTRACT. We say that a topological group G is *anti-self-dual* if there are no nontrivial continuous homomorphisms from G into the character group $\hat{G}$ or from $\hat{G}$ into G. We show that no nontrivial LCA group can be anti-self-dual.

In this note we describe the class of locally compact Abelian (LCA) groups which have no nontrivial continuous homomorphisms into their duals; in symbols, those LCA groups G for which $\mathrm{Hom}(G, \hat{G}) = \{0\}$ (see [1, 23.34] for notation). By an "anti-self-dual" topological group we shall mean one for which $\mathrm{Hom}(G, \hat{G}) = \mathrm{Hom}(\hat{G}, G) = \{0\}$. From our description we show that no nontrivial LCA group can be anti-self-dual.

A word about notation. We denote by R, Q_d, and F_p the additive group of real numbers with the usual topology, the additive group of rational numbers taken discrete, and the group of p-adic numbers (where p is a prime integer), respectively. For information on the groups F_p see [1, 10.2]. We shall also use the cyclic groups $Z(n)$ of order n and the quasicyclic groups $Z(p^\infty)$ as defined, for example, in [1, p. 3]. All groups throughout will be assumed to be LCA and Hausdorff topological groups. A group is called "densely divisible" if it has a divisible dense subgroup (see [1, A.5] for the definition of divisibility) and "reduced" if it has no nonzero divisible subgroups.

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LEMMA 1. *Every nonreduced LCA group contains a closed subgroup topologically isomorphic to either* (1) R, (2) Q_d, (3) $Z(p^\infty)$, (4) F_p, *or* (5) *a quotient of* $(Q_d)^\wedge$ *by a closed subgroup.*

PROOF. This is a rewording of [3, Theorem 6.4], taking into account [3, 2.5 and 2.7].

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LEMMA 2. *Let G be a torsion-free LCA group. If G contains a closed subgroup H topologically isomorphic to F_p, then H is a topological direct factor of G.*

PROOF. See [3, Theorem 4.21].

LEMMA 3. *Let G be a totally disconnected LCA group. Then statements* (1) *and* (2) *below are equivalent*:

(1) *G is densely divisible and $\hat{G}$ is reduced.*

(2) $\mathrm{Hom}(G, \hat{G}) = \{0\}$.

PROOF. First assume (2). Then $\hat{G}$ must be torsion-free, since if $\hat{G}$ contained a subgroup of the form $Z(n)$, where $n \geqq 2$, then $Z(n)$ would be a quotient of G, and so the projection of G onto $Z(n)$, followed by the injection of $Z(n)$ into $\hat{G}$, would yield a nontrivial member of $\mathrm{Hom}(G, \hat{G})$. Hence $\hat{G}$ is torsion-free, so G is densely divisible, by [2, Theorem 5.2].

We next show that $\hat{G}$ is totally disconnected. Assume not. Then there exists $f \neq 0$ in $\mathrm{Hom}(R, \hat{G})$ [1, 25.20]. Then the transpose map f^* is a nontrivial member of $\mathrm{Hom}(G, R)$ (see [1, 24.37]). Since G is densely divisible, f^* has dense image, so that the composition $f \circ f^*$ is a nontrivial member of $\mathrm{Hom}(G, \hat{G})$, a contradiction. Hence $\hat{G}$ is totally disconnected.

Finally, we show that $\hat{G}$ is reduced. Assume not. Then $\hat{G}$ contains as a subgroup an isomorphic copy of one of the groups (1)–(5) of Lemma 1. Now (1) and (5) are ruled out, since $\hat{G}$ is totally disconnected, while (3) is ruled out because $\hat{G}$ is torsion-free. Further, since G is totally disconnected, every element of $\hat{G}$ is compact [1, 24.17], and so (2) is ruled out. This leaves only (4). If $\hat{G}$ had a closed subgroup H topologically isomorphic to F_p, then, by Lemma 2, H would be a direct factor of $\hat{G}$. Since F_p is self-dual (see [1, 25.1]), this would imply that $\mathrm{Hom}(G, \hat{G}) \neq \{0\}$, a contradiction. We have now ruled out all cases (1)–(5). so that $\hat{G}$ must be reduced. This proves that (2) implies (1).

Conversely, if (1) holds, let D be a divisible dense subgroup of G. Since $\hat{G}$ is reduced, it follows that any f in $\mathrm{Hom}(G, \hat{G})$ must be trivial on D and hence on G, so that (1) implies (2). Q.E.D.

THEOREM. *The following two statements are equivalent for any LCA group G*:

(1) (a) *G is densely divisible,*

(b) *every element of G is compact,*

(c) *the subgroup $(\hat{G})_b$ of compact elements of $\hat{G}$ is reduced.*

(2) $\mathrm{Hom}(G, \hat{G}) = \{0\}$.

PROOF. Assume (2). As in the proof of Lemma 3, conditions (a) and (b) of (1) must hold. Let C be the identity component of G. Then the hypothesis (2) implies that $\mathrm{Hom}(G/C, (G/C)^\wedge) = [0]$. But since G/C is totally disconnected, $(G/C)^\wedge$ is reduced, by Lemma 3. Since $(G/C)^\wedge \cong (\hat{G})_b$ [1, 24.17], we have shown that (c) holds also, so that (2) implies (1).

Conversely, assume (1), and let f be in $\mathrm{Hom}(G, \hat{G})$. Since every element of G is compact, $f(G)$ is a subset of $(\hat{G})_b$. Let C be the identity component of G. Since condition (1b) implies that $\hat{G}$ is totally disconnected [1, 24.17], we conclude that $f(C)$ is just the identity of $\hat{G}$. Let u denote the natural mapping of G onto G/C and define F: $G/C \rightarrow (\hat{G})_b$ by the rule $F(u(x)) = f(x)$ for each x in G. It is easy to see that F is a well-defined homomorphism; it is also continuous, because u is an open mapping. Now G/C is totally disconnected and densely divisible, and $(G/C)^\wedge \cong (\hat{G})_b$ is reduced. Hence, by Lemma 3, $F=0$, whence it follows that $f=0$. Therefore $\mathrm{Hom}(G, \hat{G}) = \{0\}$, so that (1) implies (2). Q.E.D.

COROLLARY. *Let G be a nontrivial LCA group. Then G is not anti-self-dual.*

PROOF. We shall give a proof by contradiction. Assume that $\mathrm{Hom}(G, \hat{G}) = \mathrm{Hom}(\hat{G}, G) = \{0\}$. Since $\mathrm{Hom}(G, \hat{G}) = \{0\}$, $(\hat{G})_b$ is reduced, by the theorem. Since $\mathrm{Hom}(\hat{G}, G) = \{0\}$, $\hat{G}$ is densely divisible and $\hat{G} = (\hat{G})_b$, again by the theorem. Thus $\hat{G}$ is at once densely divisible and reduced, a contradiction. Q.E.D.

REMARK. We mention a few special cases of the theorem: If G is compact (respectively, connected), then $\mathrm{Hom}(G, \hat{G}) = \{0\}$ if and only if G is connected (respectively, compact), and if G is discrete, then $\mathrm{Hom}(G, \hat{G}) = \{0\}$ if and only if G is a weak direct sum of groups $Z(p^\infty)$ for various primes p. These statements can easily be shown directly; in fact, they provided the original motivation for the theorem.

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