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A THEOREM ON NEAR EQUICONTINUITY OF TRANSFORMATION GROUPS

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ABSTRACT. A transformation group is nearly equicontinuous if the set of nonequicontinuous points is zero dimensional and compact. It has been shown that if a transformation group is nearly equicontinuous with locally compact, locally connected metric phase space and if the set of equicontinuous points is connected, then the set N of nonequicontinuous points can contain at most two minimal sets. In this paper we will show that if in addition the phase space is not compact, then N contains exactly one minimal set.

The purpose of this paper is to prove an additional theorem related to results in [3]. This theorem shows what happens when the phase space of [3] is assumed to be not compact. This result, in particular Corollary 3, is related to the results in [5] and [6]. Notation and definitions are as in the book [2]. All topological spaces considered are assumed to be Hausdorff.

THEOREM. *Let (X, T, π) be a transformation group such that X is a locally compact, locally connected metric space which is not compact, the set E of all points of X at which T is equicontinuous is connected, and the set $N=X-E$ is zero dimensional and compact. Then N contains exactly one minimal set (that is, a nonempty set which is the orbit closure of each of its points).*

This theorem will be proved by applying the theorem in [3] to the one-point compactification of X. The following remarks, which are known for the most part, show that the hypothesis of the theorem in [3] are satisfied.

REMARK 1. Let (X, T, π) be a transformation group where X is locally compact. Then $(\bar{X}, T, \bar{\pi})$ is a transformation group where $\bar{X}$ is the one-point compactification of X and $(\bar{x}, t)\bar{\pi}=\bar{x}$ for each t in T and $(x, t)\bar{\pi}=(x, t)\pi$ for each x in X and t in T. Here $\bar{x}$ is the one point added to make X compact.

PROOF. The first two axioms are trivial. That $\bar{\pi}$ is continuous follows from 1.18 (4) of [2].

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REMARK 2. Under the assumption of the theorem X and $\tilde{X}$ are connected.

PROOF. E is dense in X and connected and the result follows.

REMARK 3. Under the assumption of the theorem $\tilde{X}$ is metrizable.

PROOF. This follows from the fact that X is connected and the following theorems [**1**, Theorems XI.7.3, XI.7.2, IX.5.6, XI.8.6].

REMARK 4. If (X, T, π) is a transformation group, if U and V are compatible uniformities of X, if $x \in X$ and if $\mathrm{cl}(xT)$ is compact, then x is equicontinuous with respect to U if and only if x is equicontinuous with respect to V.

PROOF. Use the Lesbesgue covering lemma for a compact subset of a uniform space.

REMARK 5. If X is a locally compact space, then the uniformity of its one-point compactification $\tilde{X}$ induces on X the least uniformity compatible with the topology of X.

REMARK 6. Let (X, T, π) be a transformation group where X is locally compact. Let $\tilde{E}$ be the set of equicontinuous points of $\tilde{X}$ and $\tilde{N} = \tilde{X} - \tilde{E}$. If N is compact, then $N \subset \tilde{N}$ and $E \subset \tilde{E}$.

PROOF. The first statement follows from Remark 4 and the second from Remark 5.

REMARK 7. If X is locally compact, connected and locally connected, then $\tilde{X}$ is locally connected.

PROOF. This follows from [**4**, Theorem 3.9].

REMARK 8. Under the assumption of the theorem $\tilde{x}$ is in $\tilde{N}$.

PROOF. This follows from property (4) on p. 62 of [**3**] (which shows that no equicontinuous point can be fixed).

It is easy to see from the above remarks that $(\tilde{X}, T, \tilde{\pi})$ satisfies the conditions of the theorem in [**3**], and that $\{\tilde{x}\}$ is minimal in $\tilde{N}$. Therefore N can contain at most one minimal set in order that $\tilde{N}$ will contain at most two. But [**2**, Theorem 2.22] implies N must contain at least one minimal set. Therefore, N contains exactly one minimal set.

In the following corollaries the hypothesis of the theorem is assumed.

COROLLARY 1. *If x_1 and x_2 belong to X, then* $\mathrm{cl}(x_1T) \cap \mathrm{cl}(x_1T)$ *is not empty.*

PROOF. If $x \in E$, then we know from [**3**] that $\mathrm{cl}(xT) \supset N$. If $x \in N$, then we know from [**2**, Theorem 2.22] that $\mathrm{cl}(xT)$ contains a minimal set. In either case $\mathrm{cl}(xT)$ must contain the minimal set of N.

COROLLARY 2. *If T is almost periodic at each point of N, then N is minimal.*

COROLLARY 3. *If T is abelian or connected, then N contains exactly one point and this point is fixed under T.*

Corollaries 2 and 3 follow in the same way as Corollaries 2 and 3 of [3].

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