COMPLEX BORDISM OF CLASSIFYING SPACES

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ABSTRACT. It is shown that, for a finite group $G$, the Thom homomorphism maps the complex bordism $\Omega_*^U(B_G)$ onto the homology $H_*(B_G; \mathbb{Z})$ of a classifying space for $G$ if and only if $G$ has periodic cohomology.

Let $B_G$ denote a classifying space for the finite group $G$, and let

$$\mu: \Omega_*^U(B_G) \to H_*(B_G; \mathbb{Z})$$

denote the Thom homomorphism ([5], [7]) from the complex bordism of $B_G$ to its integral homology. It is well known (e.g. see [9]) that $\mu$ is onto if $G$ is cyclic, and more generally if $G$ has periodic cohomology [4, Chapter XII, §11], since then $H^n(B_G; \mathbb{Z}) = 0$ for all odd $n$, thus $H_*(B_G, \mathbb{Z}) = 0$ for all even $p > 0$ and this makes the complex bordism spectral sequence collapse. The main result of this note is the converse statement.

**Theorem 1.** If $G$ is a finite group for which $\mu: \Omega_*^U(B_G) \to H_*(B_G; \mathbb{Z})$ is onto, then $G$ has periodic cohomology.

The main step in the proof of this result has recently been supplied by R. G. Swan [11].

**Theorem 2.** A finite group $G$ with $H^n(B_G; \mathbb{Z}) = 0$ for all odd $n$ must have periodic cohomology.

We state the next result in order to give a more complete picture. For a CW-complex $X$ write $\text{dim} \Omega_*^U(X)$ for the projective dimension of the $\Omega_*^U$-module $\Omega_*^U(X)$ [7].

**Theorem 3.** The following conditions are equivalent for a finite group $G$:

(a) $G$ has periodic cohomology, i.e. every abelian subgroup of $G$ is cyclic [4, Chapter XII, §11].
(b) $H^n(B_G; \mathbb{Z}) = 0$ for all odd $n$.
(c) The Atiyah spectral sequence [2] $H^*(B_G; \mathbb{Z}) \Rightarrow K^*(B_G) = (R(G))^\wedge$ collapses.

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(d) The Thom homomorphism $\mu : \Omega^U_*(B_\theta) \to H_*(B_\theta; \mathbb{Z})$ is onto.
(e) $\text{hom dim}_{\mathbb{Q}} \Omega^U_*(B_\theta) \leq 1$.

The implication (a) $\implies$ (b) follows by reduction to the Sylow subgroups and a computation [4, Chapter XII]. The implications (b) $\implies$ (c) and (b) $\implies$ (d) are well-known consequences of the vanishing of the even differentials in the $K$-theory and complex bordism spectral sequences. Furthermore the implication (c) $\implies$ (b) follows from the facts [2] that $K^1(B_\theta) = 0$ and that the $K$-theory filtration is even. The equivalence of (d) and (e) follows from Proposition 4 below.

We turn to the proof of Theorem 1. If the Thom homomorphism maps $\Omega^U_*(B_\theta)$ onto $H_*(B_\theta; \mathbb{Z})$, then the complex bordism spectral sequence

$$H_*(B_\theta; \Omega^U_*) \Rightarrow \Omega^U_*(B_\theta)$$

(1)

collapses. Let $K_*(\cdot)$ denote the $\mathbb{Z}_2$-graded homology theory dual to $K$-theory. J. W. Vick [12] has proved that $K_1(B_\theta)$ is isomorphic to the character group of the compact group $K_0(B_\theta)$, and that $K_0(B_\theta) = 0$. There is the $K$-spectral sequence

$$H_*(B_\theta; \mathbb{Z}) \Rightarrow K_*(B_\theta),$$

(2)

and we claim that it collapses if (1) does.

The natural transformation $\mu_* : \Omega^U_*(\cdot) \to K_*(\cdot)$ dual to the corresponding transformation from complex cobordism to $K$-theory [6] induces a morphism of the spectral sequence (1) into the spectral sequence (2). Since on the coefficient rings $\mu_*$ is the Todd genus and so maps $\Omega^U_*$ onto $K_*(pt) = \mathbb{Z}$, (2) indeed collapses if (1) does.

In the spectral sequence (2) we have $E^2_p = H_p(B_\theta; \mathbb{Z})$ and

$$E^\infty_p \cong \text{Im}(K_p(B_\theta) \to K_p(B_\theta))/\text{Im}(K_p(B_\theta^{p-1}) \to K_p(B_\theta)),$$

where $B^{k}_\theta$ is the $k$-skeleton of the complex $B_\theta$, and thus $E^\infty_p = 0$ for even $p > 0$. If (2) collapses then $E^2 = E^\infty$ and so $H_p(B_\theta; \mathbb{Z}) = 0$ for all even $p > 0$. By the universal coefficient theorem it follows that $H^n(B_\theta; \mathbb{Z}) = 0$ for all odd $n$, and then Theorem 1 is a direct consequence of Theorem 2.

REMARK 1. A rapid but indirect proof that a finite $p$-group $G$ with $H^n(B_\theta; \mathbb{Z}) = 0$ for all odd $n$ must have periodic cohomology can be based on Lemma (2.1) and Proposition (6.1) of [10]. Then $G$ is either a cyclic or generalized quaternion group.

REMARK 2. The same argument shows that the Thom homomorphism $\mu : \Omega^*_U(B_\theta) \to H^*(B_\theta; \mathbb{Z})$ from complex cobordism to cohomology
is onto if and only if \( G \) has periodic cohomology. In fact, if \( X \) is any complex and if either of the spectral sequences

\[
H^*(X; \mathbb{Z}) \Rightarrow K^*(X), \quad H^*(X; \Omega^*) \Rightarrow \Omega^*(X)
\]
collapses, then so does the other. This observation is due to T. tom Dieck [8, §6], and requires the theorem of Hattori and Stong; the homology analogue is also true.

The equivalence of (d) and (e) in Theorem 3 is a consequence of the next result.

**Proposition 4.** For any CW-complex \( X, \mu: \Omega^u(X) \to H_*(X; \mathbb{Z}) \) is onto if and only if \( \text{hom dim} \Omega^u_*(X) \leq 1 \).

For finite complexes the proof is given in [7, §3]. The following argument was suggested by Professor Larry Smith for the general case. We may regard \( X \) as a CW-spectrum in J. M. Boardman’s stable category [3], and then \( X \) is \((-1)\)-connected.

**Lemma 5.** Let \( X \) be an \( n \)-connected CW-spectrum. Then there exists a cofibration \( A_1 \to A_0 \to X \) of \( n \)-connected CW-spectra such that \( \Omega^u_*(A_0) \) is a free \( \Omega^u_* \)-module and the sequence of induced homomorphisms

\[
0 \to \Omega^u_*(A_1) \to \Omega^u_*(A_0) \to \Omega^u_*(X) \to 0
\]

is exact.

With the lemma in hand, it is entirely straightforward to show that the arguments of [7, §3] for finite complexes carry over to highly-connected CW-spectra and thus apply to all CW-complexes.

It remains to prove the lemma, and there is no loss of generality in supposing that \( X \) is \((-1)\)-connected. Then the proof can be easily extracted from Lemmas 18 and 19 and Example (iv) on p. 29 of [1, Lecture 1]. It is only necessary to insist in Lemma 18 that the finite subcomplex \( X' \) have dimension \( \leq p \), and to then choose \( E_a \) to be the \( p \)-skeleton of \( E = MU \); then \( Sp \wedge DE_a \) has the homotopy type of a finite complex of dimension \( \leq p \), and so is \((-1)\)-connected. Now Lemma 19 yields the desired cofibration.

**Remark 3.** For a finite group \( G \), let \( \text{rank}(G) \) denote the largest integer \( k \) such that \( G \) contains an abelian subgroup with minimum number of generators \( k \). I conjecture that

\[
(3) \quad \text{hom dim} \Omega^u_*(BG) = \text{rank} (G).
\]

For example, we know that \( \text{rank}(G) = 1 \) if and only if \( \text{hom} \)
\[ \dim_\mathbb{Q} \Omega^U_*(B) = 1. \] (If \( \Omega^U_*(B) \) is a projective module then \( H_*(B; \mathbb{Z}) \) is free abelian, hence \( \tilde{H}_*(B; \mathbb{Z}) = 0 \) and so \( G \) is trivial.) If \( G \) is elementary abelian the results of [7, §5] show that
\[ \text{hom dim}_\mathbb{Q} \Omega^U_*(B) \geq \text{rank}(G), \]
but even in this case it is not generally known if \( \Omega^U_*(B) \) has finite projective dimension.

Our final result concerns the relation between \( \Omega^U_*(B) \) and \( \Omega^U_*(B_H) \) when \( H \) is a Sylow \( p \)-subgroup of \( G \). In this situation there is the following diagram of induced homomorphisms and transfer homomorphisms [5, §20]:
\[
\begin{array}{ccc}
\Omega_n^U(B) & \xrightarrow{t} & \Omega_n^U(B_H) & \xrightarrow{i} & \Omega_n^U(B) \\
\downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
H_n(B; \mathbb{Z}) & \xrightarrow{t} & H_n(B_H; \mathbb{Z}) & \xrightarrow{i} & H_n(B; \mathbb{Z}).
\end{array}
\]
Let \( \Omega^U_n(B)_{(p)} \) and \( H_n(B; \mathbb{Z})_{(p)} \) denote the \( p \)-primary parts. It is well known that on \( H_n(B; \mathbb{Z}) \) the composition \( i \circ t \) is multiplication by the index of \( H \) in \( G \). Since this index is prime to \( p \), it follows that \( \tilde{H}_n(B_H; \mathbb{Z}) \to H_n(B; \mathbb{Z}) \) is a split epimorphism.

**Proposition 6.** Let \( H \) be a Sylow \( p \)-subgroup of the finite group \( G \). Then \( i : \Omega^U_*(B_H) \to \Omega^U_*(B)_{(p)} \) is a split epimorphism of \( \Omega^U_* \)-modules.

**Proof.** It suffices to show that \( i \circ t \) is an automorphism of \( \Omega^U_*(B)_{(p)} \). Both \( i \) and \( t \) induce morphisms of spectral sequences, yielding the following composition on the \( E^2 \)-terms:
\[
\begin{align*}
H_*(B_G; \Omega^U_*) & \xrightarrow{t} H_*(B_H; \Omega^U_*) \xrightarrow{i} H_*(B_G; \Omega^U_*) \\
\end{align*}
\]
Since this restricts to an isomorphism on the \( p \)-primary parts, it follows that \( i \circ t \) yields an isomorphism on the \( p \)-primary parts of the \( E^\infty \)-terms. Since the complex bordism spectral sequence is convergent, it follows that \( i \circ t \) is an automorphism of \( \Omega^U_*(B)_{(p)} \).

**Remark 4.** It follows that the projective dimension of the \( \Omega^U_* \)-module \( \Omega^U_*(B) \) is bounded by the maximum of the projective dimensions of the modules \( \Omega^U_*(B_H) \) as \( H \) runs through the Sylow subgroups of \( G \). Of course the rank of \( G \) is the maximum of the ranks of the Sylow subgroups of \( G \). If the conjecture (3) holds for \( p \)-groups, and if
\[ \text{hom dim}_\mathbb{Q} \Omega^U_*(B_H) \leq \text{hom dim}_\mathbb{Q} \Omega^U_*(B) \]
whenever $H$ is a subgroup of $G$, then the conjecture is true in general. I would very much like to know an upper bound to $\text{hom dim}_{\mathbb{Z}}(B_\sigma)$ for $p$-groups.

References