REAL ZEROS OF A RANDOM SUM
OF ORTHOGONAL POLYNOMIALS

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Abstract. Let $c_0, c_1, c_2, \cdots$ be a sequence of normally distributed independent random variables with mathematical expectation zero and variance unity. Let $P_k(x)$ ($k = 0, 1, 2, \cdots$) be the normalised Legendre polynomials orthogonal with respect to the interval $(-1, 1)$. It is proved that the average number of the zeros of $c_0P_0^*(x) + c_1P_1^*(x) + \cdots + c_nP_n^*(x)$ in the same interval is asymptotically equal to $(3)^{-1/2}n$ when $n$ is large.

1. Let $\phi_0(x), \phi_1(x), \phi_2(x), \cdots$ be a sequence of polynomials orthogonal with respect to a given positive-valued weight function $\omega(x)$ over the interval $(a, b)$ where one or both of $a$ and $b$ may be infinite and let $\psi_n(x) = \frac{1}{g_n} \phi_n(x)$ with

$$g_n = \int_a^b \omega(x) \phi_n^2(x) dx.$$

Let $f(x)$ be defined by

$$f(x) = f(c; x) = \sum_{k=0}^N c_k \psi_k(x),$$

where the coefficients $c_0, c_1, c_2, \cdots$ form a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance unity. We take the ordered set $c_0, \cdots, c_n$ as the point $c$ in an $(n+1)$-dimensional real vector space $R_{n+1}$. The probability that the point $c$ lies in an "infinitesimal rectangle" $\Pi(c)$ with sides of lengths $dc_0, dc_1, \cdots, dc_n$ is

$$dP(c) = \prod_{k=0}^n \left\{ (2\pi)^{-1/2} \exp(-\frac{1}{2}c_k^2) dc_k \right\}.$$

Let $N(c; \alpha, \beta)$ denote the number of zeros of the polynomial (1.1) in the interval $\alpha \leq x \leq \beta$. We establish the formula

$$\int_{R_{n+1}} N(c; \alpha, \beta) dP(c) = \frac{1}{\pi} \int_{\alpha}^{\beta} \left[ \frac{S_n(x) + R_n(x)}{D_n(x)} - \frac{1}{4} \frac{Q_n^2(x)}{D_n^2(x)} \right] dx,$$
where
\[ D_n(x) = \phi_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi_n'(x), \]
\[ Q_n(x) = \phi_{n+1}''(x)\phi_n(x) - \phi_{n+1}''(x)\phi_n'(x), \]
\[ R_n(x) = \frac{1}{2} \left\{ \phi_{n+1}''(x)\phi_n'(x) - \phi_{n+1}(x)\phi_n''(x) \right\} \]

and
\[ S_n(x) = \frac{1}{2} \left\{ \phi_{n+1}''(x)\phi_n'(x) - \phi_{n+1}(x)\phi_n''(x) \right\}. \]

When \( n \) is large, we can find an estimate of the integrand in the right-hand side of (1.2) in terms of \( n \) and \( x \) only in an easily integrable form, since only two functions \( \phi_n(x) \) and \( \phi_{n+1}(x) \) are now involved.

Let \( P_k^\ast(x) \) be the normalized Legendre polynomial \((k + \frac{1}{2})^{1/2}P_k(x)\), where
\[ P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k; \]

the famous Legendre polynomial. Here \( a = -1 \) and \( b = 1 \) and \( \omega(x) = 1 \). Further \( \psi_k(x) = P_k^\ast(x) \) with \( g_n = (n + \frac{1}{2})^{1/2} \). We prove

**Theorem 1.** The average number of zeros of
\[ c_0P_0^\ast(x) + c_1P_1^\ast(x) + \cdots + c_kP_k^\ast(x) + \cdots + c_nP_n^\ast(x) \]
in \((-1, 1)\) is asymptotically equal to \( n/\sqrt{3} \) when \( n \) is sufficiently large.

2. Let us put
\[ A = A_n(x) = \psi_0^2(x) + \psi_1^2(x) + \cdots + \psi_n^2(x), \]
\[ B = B_n(x) = \psi_1(x)\psi_1'(x) + \cdots + \psi_n(x)\psi_n'(x) \]
and
\[ C = C_n(x) = [\psi_1'(x)]^2 + \cdots + [\psi_n'(x)]^2. \]

Then, by Cauchy’s inequality, \( AC - B^2 \geq [\psi_0\psi_1']^2 > 0 \). By proceeding as in §3 of our earlier work (cf. [1]), we obtain
\[ \int_{R_{n+1}} N(c; \alpha, \beta) dP(c) = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{(AC - B^2)^{1/2}}{A} dx. \]

We put \( \lambda_n = h_n g_n^{-1} h_{n+1}^{-1} \), where \( h_n \) is the coefficient of \( x^n \) in \( \phi_n(x) \) and \( g_n \) is defined as above. Then we have
\[ \sum_{k=0}^{n} g_k^{-1} \phi_k(x)\phi_k(y) = \lambda_n \frac{\phi_{n+1}(y)\phi_n(x) - \phi_{n+1}(x)\phi_n(y)}{y - x}. \]

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This is the famous Christofel-Darboux formula [3, p. 135] in the theory of orthogonal functions. We set $y = x + \delta$ in the formula (2.2) and equate the coefficients of like powers of $\delta$ on both sides to obtain

\begin{align*}
(2.3) \quad \sum_{r=0}^{n} g_{r}^{-1} [\phi_{r}(x)]^2 &= \lambda_{n} [\phi_{n+1}(x) \phi_{n}(x) - \phi_{n+1}(x) \phi'_{n}(x)], \\
(2.4) \quad \sum_{r=1}^{n} g_{r}^{-1} [\phi_{r}(x) \phi'_{r}(x)] &= \frac{\lambda_{n}}{2} [\phi_{n+1}(x) \phi_{n}(x) - \phi_{n+1}(x) \phi''_{n}(x)],
\end{align*}

and

\begin{align*}
(2.5) \quad \sum_{r=1}^{n} g_{r}^{-1} [\phi_{r}(x) \phi''_{r}(x)] &= \frac{\lambda_{n}}{3} [\phi_{n+1}(x) \phi_{n}(x) - \phi_{n+1}(x) \phi'''_{n}(x)].
\end{align*}

Differentiating (2.4) and making use of (2.5), we get

\begin{align*}
(2.6) \quad \sum_{r=1}^{n} g_{r}^{-1} [\phi'_{r}(x)]^2 &= \frac{\lambda_{n}}{6} [\phi''_{n+1}(x) \phi_{n}(x) - \phi_{n+1}(x) \phi'''_{n}(x)] \\
&+ \frac{\lambda_{n}}{2} [\phi''_{n+1}(x) \phi'_{n}(x) - \phi_{n+1}(x) \phi''_{n}(x)].
\end{align*}

Making use of (2.3), (2.4) and (2.6), and the fact that $\lambda_{n} \neq 0$, we obtain the formula (1.2).

3. For Legendre polynomials $P_{n}(x)$, we have the relations

\begin{align*}
(3.1) \quad (1 - x^{2}) P_{n+1}'(x) &= 2x P_{n+1}'(x) - (n + 1)(n + 2) P_{n+1}(x) \\
(3.2) \quad (1 - x^{2}) P_{n}'(x) &= 2x P_{n}'(x) - n(n + 1) P_{n}(x).
\end{align*}

From (3.1) and (3.2), we obtain

\begin{align*}
(3.3) \quad (1 - x^{2}) [P_{n+1}'(x) P_{n}'(x) - P_{n}'(x) P_{n+1}(x)] &= - (n + 1) \{ n \{ P_{n+1}(x) P_{n}'(x) - P_{n}(x) P_{n+1}'(x) \} + 2 P_{n+1}(x) P_{n}'(x) \} \\
&+ 2x [P_{n+1}'(x) P_{n}(x) - P_{n+1}(x) P_{n}'(x)].
\end{align*}

Differentiating (3.4) and using (3.3), we get
(1 - x^2)\[P''_{n+1}(x)P_n(x) - P''_n(x)P_{n+1}(x)\]
= (n + 1)\left[n\left\{P_{n+1}(x)P'_n(x) - P_n(x)P'_{n+1}(x)\right\} + 2P_{n+1}(x)P'_n(x)\right]
+ \frac{16}{1 - x^2} \left[x\left\{P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)\right\}
- (n + 1)P_n(x)P_{n+1}(x)\right]
+ 2(n + 1)\left[P'_n(x)P_{n+1}(x) - P_n(x)P'_{n+1}(x)\right].

We recall another formula for the derivative of a Legendre function [2, p. 179, (17)], viz.:

\begin{align}
(3.6) \quad (x^2 - 1)P'_n(x) &= nxP_n(x) - nP_{n-1}(x) \\
(3.7) \quad (x^2 - 1)P'_{n+1}(x) &= (n + 1)xP_{n+1}(x) - (n + 1)P_n(x).
\end{align}

The application of (3.6) and (3.7) yields

\begin{align}
(3.8) \quad (x^2 - 1)\left[P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)\right]
= (n + 1)\left[2xP_n(x)P_{n+1}(x) - P^2_n(x) - P^2_{n+1}(x)\right],

(3.9) \quad (x^2 - 1)\left[P'_{n+1}(x)P_n(x) + P_{n+1}(x)P'_n(x)\right]
= (n + 1)\left[P^2_{n+1}(x) - P^2_n(x)\right]
\end{align}

and

\begin{align}
(3.10) \quad (x^2 - 1)P_{n+1}(x)P'_n(x) &= (n + 1)P_{n+1}(x)\left[P_{n+1}(x) - xP_n(x)\right].
\end{align}

To evaluate

\[P^2_n(x) + P^2_{n+1}(x) - 2xP_n(x)P_{n+1}(x),\]

we set \(x = \cos \gamma\) and make use of the celebrated Laplace’s formula (cf. [2, p. 208]) giving the asymptotic value of \(P_n(\cos \gamma)\) as

\[\left(\frac{2}{\pi n \sin \gamma}\right)^{1/2} \cos \left[\left(n + \frac{1}{2}\right)\gamma - \frac{\pi}{4}\right] + O((n \sin \gamma)^{-3/2})\]

in the range \(\epsilon < \gamma < \pi - \epsilon\), where \(0 < \epsilon < \pi/2\). After some simplifications, we find
\[ P_n^2(x) + P_{n+1}^2(x) - 2xP_n(x)P_{n+1}(x) \]
\[
= \frac{2}{\pi n \sin \gamma} \left\{ \cos^2 \left[ \left( n + \frac{1}{2} \right) \gamma - \frac{\pi}{4} \right] + \cos^2 \left[ \left( n + \frac{3}{2} \right) \gamma - \frac{\pi}{4} \right] \right. \\
- 2 \cos \gamma \cos \left[ \left( n + \frac{1}{2} \right) \gamma - \frac{\pi}{4} \right] \cos \left[ \left( n + \frac{3}{2} \right) \gamma - \frac{\pi}{4} \right] \left\} + O(n^{-2} \csc^2 \gamma) \right.
\]
\[
= \frac{2}{\pi n} (1 - x^2)^{1/2} + O(n^{-2}(1 - x^2)^{-1}).
\]

Making use of (3.8), we obtain

\[ \{ P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x) \} > \frac{2}{\pi} (1 - x^2)^{-1/2} \]

for sufficiently large \( n \) and \( |x| < 1 - n^{-2/3} \log n \). By the first theorem of Stieltjes, [2, p. 197, (8)] \( |P_n(x)| \leq 4n^{-1/2}(1 - x^2)^{-1/4} \) and by (3.6), \( |P'_n(x)| \leq 8n^{1/2}(1 - x^2)^{-5/4} \). Thus

\[ nP_n(x)P_{n+1}(x) = O((1 - x^2)^{-1/2}), \]

\[ P_n(x)P'_n(x) = O((1 - x^2)^{-3/2}) \]

and

\[ P'_{n+1}(x)P_n(x) + P'_n(x)P_{n+1}(x) = O((1 - x^2)^{-3/2}). \]

By putting these estimates in (3.3), (3.4) and (3.5), we get

\[ (1 - x)(P''_{n+1}P'_n - P''_nP_{n+1}) \]
\[
= n(n + 1)(P'_{n+1}P_n - P'_nP_{n+1}) + O(n(1 - x^2)^{-3/2}), \]
\[
(1 - x)(P''_{n+1}P_n - P''_nP_{n+1}) \]
\[
= 2x(P'_{n+1}P_n - P'_nP_{n+1}) + O((1 - x^2)^{-3/2}) \]

and

\[ (1 - x)(P''_{n+1}P_n - P''_nP_{n+1}) = \left\{ \frac{16x}{1 - x^2} - n - \frac{n^3}{2} \right\} (P'_{n+1}P_n - P'_nP_{n+1}) \]
\[
+ O(n(1 - x^2)^{-3/2}), \]
where we have written $P_k$, $P'_k$, $P''_k$ and $P'''_k$ for $P_k(x)$, $P'_k(x)$, $P''_k(x)$ and $P'''_k(x)$, respectively. This abbreviation is also employed below.

By using (3.11), we finally obtain, for $|x| < 1 - n^{-2/3} \log n$, the estimate

$$\frac{(P_{n+1}P_n - P_{n+1}P''_n)}{(P_{n+1}P_n - P_{n+1}P'_{n})} = - n^2 (1 - x^2)^{-1} (1 + O(1/n)),$$

$$\frac{(P_{n+1}P'_{n} - P_{n+1}P''_{n})}{(P_{n+1}P_n - P_{n+1}P'_{n})} = n^2 (1 - x^2)^{-1} (1 + O(1/n))$$

and

$$\frac{(P_{n+1}P_n - P_{n+1}P''_n)}{(P_{n+1}P_n - P_{n+1}P'_{n})} = O(n^2 (1 - x^2)^{-1}).$$

Putting these values for $Q'(x)/Q(x)$, $P'(x)/P(x)$ and $S'(x)/S(x)$ in (1.2), the expression enclosed by brackets is estimated by

$$\frac{n^2}{3} \frac{1}{(1 - x^2)} \left[ 1 + O\left(\frac{1}{(\log n)^{3/2}}\right) \right].$$

Let $\epsilon = n^{-2/3} \log n$ and $N(c; \epsilon)$ denote the number of zeros of

$$f(c; x) = c_0 P_0^*(x) + c_1 P_1^*(x) + \cdots + c_n P_n^*(x)$$

in $-1 + \epsilon \leq x \leq 1 - \epsilon$. By (1.2), we have

$$\int_{R_{n+1}} N(c; \epsilon) d\lambda(c) = \frac{1}{\pi} \int_{-1+\epsilon}^{1-\epsilon} \frac{n}{\sqrt{3}} [1 + O((\log n)^{-3})](1 - x^2)^{-1/2} dx$$

$$= \frac{n}{\sqrt{3}} \{1 + O((\log n)^{-3})\}.$$

To complete the proof of Theorem 1, we observe (cf. [2, p. 250]) that

$$|P_n(z)| = \left| \frac{1}{\pi} \int_0^{2\pi} \{z + i(1 - z^2)^{1/2} \cos \tau\}^n d\tau \right|$$

and thus for $z = 1 + \epsilon e^{i\theta}$, $|P_n(z)| < (1 + 3\epsilon)^n < 2n^n \exp(n^{1/3})$. Further $P_n(1) = 1$. We can prove, as in our work [1, p. 722], that

$$\Pr\left( \max_{0 \leq k \leq n} |c_k| \leq n \right) > 1 - e^{-n^{1/3}},$$

so that for $z = 1 + \epsilon e^{i\theta}$, we have

$$\Pr(|f(c; z)| \geq 4n^3 \exp(n^{1/3})) < \exp(-n^2/3)$$

and
Let \( \nu(c; \epsilon) \) denote the number of zeros of \( |f(c; z)| \) in \( |z - 1| \leq \epsilon \). By making use of Jensen's theorem, we find
\[
\nu(c; \epsilon) \log 2 \leq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(c; 1 + \epsilon e^{i\theta})}{f(c; 1)} \right| \, d\theta = O(n^{1/3})
\]
with probability at least equal to \( 1 - 2/n \). This shows that the average number of zeros of \( f(\cdot; x) \) in \( 1 - \epsilon \leq x \leq 1 \) and similarly in \( -1 \leq x \leq -1 + \epsilon \) is \( O(n^{1/3}) \). Therefore, on using (3.16), we finally obtain the proof of Theorem 1.

References


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