REAL ZEROS OF A RANDOM SUM OF ORTHOGONAL POLYNOMIALS

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Abstract. Let \( c_0, c_1, c_2, \ldots \) be a sequence of normally distributed independent random variables with mathematical expectation zero and variance unity. Let \( P_k(x) \) \((k = 0, 1, 2, \ldots)\) be the normalised Legendre polynomials orthogonal with respect to the interval \((-1, 1)\). It is proved that the average number of the zeros of \( c_0P_0(x) + c_1P_1(x) + \cdots + c_nP_n(x) \) in the same interval is asymptotically equal to \((3)^{-1/2}n\) when \( n \) is large.

1. Let \( \phi_0(x), \phi_1(x), \phi_2(x), \ldots \) be a sequence of polynomials orthogonal with respect to a given positive-valued weight function \( \omega(x) \) over the interval \((a, b)\) where one or both of \( a \) and \( b \) may be infinite and let \( \psi_n(x) = g_n^{-1/2} \phi_n(x) \) with

\[
g_n = \int_a^b \omega(x) \phi_n^2(x) dx.
\]

Let \( f(x) \) be defined by

\[
f(x) = f(c; x) = \sum_{k=0}^N c_k \psi_k(x),
\]

where the coefficients \( c_0, c_1, c_2, \ldots \) form a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance unity. We take the ordered set \( c_0, c_1, \ldots, c_n \) as the point \( c \) in an \((n+1)\)-dimensional real vector space \( R_{n+1} \). The probability that the point \( c \) lies in an "infinitesimal rectangle" \( \Pi(c) \) with sides of lengths \( dc_0, dc_1, \ldots, dc_n \) is

\[
dP(c) = \prod_{k=0}^n \{ (2\pi)^{-1/2} \exp(-\frac{1}{2}c_k^2) dc_k \}.
\]

Let \( N(c; \alpha, \beta) \) denote the number of zeros of the polynomial (1.1) in the interval \( \alpha \leq x \leq \beta \). We establish the formula

\[
(1.2) \int_{R_{n+1}} N(c; \alpha, \beta) dP(c) = \frac{1}{\pi} \int_\alpha^\beta \left[ \frac{S_n(x) + R_n(x)}{D_n(x)} - \frac{1}{4} \frac{Q_n^2(x)}{D_n^2(x)} \right]^{1/2} dx,
\]
where
\[ D_n(x) = \phi'_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'_n(x), \]
\[ Q_n(x) = \phi''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi''_n(x), \]
\[ R_n(x) = \frac{1}{2}\left\{ \phi''_{n+1}(x)\phi'_n(x) - \phi'_{n+1}(x)\phi''_n(x) \right\}, \]
and
\[ S_n(x) = \frac{1}{2}\left\{ \phi'''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'''_n(x) \right\}. \]

When \( n \) is large, we can find an estimate of the integrand in the right-hand side of (1.2) in terms of \( n \) and \( x \) only in an easily integrable form, since only two functions \( \phi_n(x) \) and \( \phi_{n+1}(x) \) are now involved.

Let \( P^*_k(x) \) be the normalized Legendre polynomial \((k + \frac{1}{2})^{1/2}P_k(x)\), where
\[ P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k; \]
the famous Legendre polynomial. Here \( a = -1 \) and \( b = 1 \) and \( \omega(x) \equiv 1 \).

Further, \( \psi_k(x) = P^*_k(x) \) with \( g_n = (n + \frac{1}{2})^{1/2} \). We prove

**Theorem 1.** The average number of zeros of
\[ c_0P^*_0(x) + c_1P^*_1(x) + \cdots + c_kP^*_k(x) + \cdots + c_nP^*_n(x) \]
in \((-1, 1)\) is asymptotically equal to \( n/\sqrt{3} \) when \( n \) is sufficiently large.

2. Let us put
\[ A = A_n(x) = \psi_0^2(x) + \psi_1^2(x) + \cdots + \psi_n^2(x), \]
\[ B = B_n(x) = \psi_1(x)\psi'_1(x) + \cdots + \psi_n(x)\psi'_n(x) \]
and
\[ C = C_n(x) = [\psi'_1(x)]^2 + \cdots + [\psi'_n(x)]^2. \]

Then, by Cauchy's inequality, \( AC - B^2 \geq [\psi_0\psi'_1]^2 > 0 \). By proceeding as in §3 of our earlier work (cf. [1]), we obtain
\[ \int_{R_{n+1}} N(c; \alpha, \beta) dP(c) = \frac{1}{\pi} \int_\alpha^\beta \frac{(AC - B^2)^{1/2}}{A} dx. \]

We put \( \lambda_n = h_ng_n^{-1}h_{n+1}^{-1} \), where \( h_n \) is the coefficient of \( x^n \) in \( \phi_n(x) \) and \( g_n \) is defined as above. Then we have
\[ \sum_{k=0}^n g_k^{-1} \phi_k(x)\phi_k(y) = \lambda_n \frac{\phi_{n+1}(y)\phi_n(x) - \phi_{n+1}(x)\phi_n(y)}{y - x}. \]
This is the famous Christofel-Darboux formula [3, p. 135] in the theory of orthogonal functions. We set $y = x + \delta$ in the formula (2.2) and equate the coefficients of like powers of $\delta$ on both sides to obtain

\begin{align}
(2.3) \quad \sum_{r=0}^{n} g_r^{-1} [\phi_r(x)]^2 &= \lambda_n [\phi_{n+1}(x) \phi_n(x) - \phi_{n+1}(x) \phi'_n(x)], \\
(2.4) \quad \sum_{r=1}^{n} g_r^{-1} [\phi_r(x) \phi'_n(x)] &= \frac{\lambda_n}{2} [\phi_{n+1}(x) \phi_n(x) - \phi_{n+1}(x) \phi''_n(x)] \\
\text{and} \\
(2.5) \quad \sum_{r=1}^{n} g_r^{-1} [\phi_r(x) \phi''_n(x)] &= \frac{\lambda_n}{3} [\phi_{n+1}(x) \phi_n(x) - \phi_{n+1}(x) \phi''_n(x)].
\end{align}

Differentiating (2.4) and making use of (2.5), we get

\begin{align}
(2.6) \quad \sum_{r=0}^{n} g_r^{-1} [\phi'_r(x)]^2 &= \frac{\lambda_n}{6} [\phi_{n+1}'(x) \phi_n(x) - \phi_{n+1}(x) \phi'''_n(x)] \\
&+ \frac{\lambda_n}{2} [\phi_{n+1}(x) \phi'_n(x) - \phi_{n+1}(x) \phi''_n(x)].
\end{align}

Making use of (2.3), (2.4) and (2.6), and the fact that $\lambda_n \neq 0$, we obtain the formula (1.2).

3. For Legendre polynomials $P_n(x)$, we have the relations

\begin{align}
(3.1) \quad (1 - x^2) P_{n+1}''(x) &= 2x P_{n+1}'(x) - (n + 1)(n + 2) P_{n+1}(x) \\
(3.2) \quad (1 - x^2) P_n'(x) &= 2x P_n'(x) - n(n + 1) P_n(x).
\end{align}

From (3.1) and (3.2), we obtain

\begin{align}
(3.3) \quad (1 - x^2) [P_{n+1}'(x) P_n'(x) - P_n'(x) P_{n+1}'(x)] &= - (n + 1) \{ n \{ P_{n+1}(x) P_n'(x) - P_n(x) P_{n+1}'(x) \} + 2 P_{n+1}(x) P_n'(x) \}
\end{align}

and

\begin{align}
(3.4) \quad (1 - x^2) [P_{n+1}'(x) P_n(x) - P_{n+1}(x) P_n'(x)] &= 2x \{ P_{n+1}'(x) P_n(x) - P_{n+1}(x) P_n'(x) \} - 2(n + 1) P_n(x) P_{n+1}(x).
\end{align}

Differentiating (3.4) and using (3.3), we get
\[(1 - x^2)[P''_{n+1}(x)P_n(x) - P''_n(x)P_{n+1}(x)]\]
\[= (n + 1)[n\{P_{n+1}(x)P'_n(x) - P_n(x)P'_{n+1}(x)\} + 2P_{n+1}(x)P'_n(x)] \]
\[+ \frac{16}{1 - x^2} [x\{P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)\}]
\[- (n + 1)P_n(x)P_{n+1}(x)] \]
\[+ 2(n + 1)[P'_n(x)P_{n+1}(x) - P_n(x)P'_{n+1}(x)].\]

We recall another formula for the derivative of a Legendre function [2, p. 179, (17)], viz.:
\[(x^2 - 1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)\]
and
\[(x^2 - 1)P'_{n+1}(x) = (n + 1)xP_{n+1}(x) - (n + 1)P_n(x).\]

The application of (3.6) and (3.7) yields
\[(x^2 - 1)[P'_{n+1}(x)P_n(x) - P_n(x)P'_{n+1}(x)]\]
\[= (n + 1)[2xP_n(x)P_{n+1}(x) - P^2_n(x) - P^2_{n+1}(x)],\]
\[(x^2 - 1)[P'_{n+1}(x)P_n(x) + P_n(x)P'_{n+1}(x)]\]
\[= (n + 1)[P^2_{n+1}(x) - P^2_n(x)]\]

and
\[(x^2 - 1)P_{n+1}(x)P'_n(x) = (n + 1)P_{n+1}(x)\{P_{n+1}(x) - xP_n(x)\}.\]

To evaluate
\[P^2_n(x) + P^2_{n+1}(x) - 2xP_n(x)P_{n+1}(x),\]
we set \(x = \cos \gamma\) and make use of the celebrated Laplace's formula (cf. [2, p. 208]) giving the asymptotic value of \(P_n(\cos \gamma)\) as
\[\left(\frac{2}{\pi n \sin \gamma}\right)^{1/2} \cos \left[\left(n + \frac{1}{2}\right)\gamma - \frac{\pi}{4}\right] + O((n \sin \gamma)^{-3/2})\]
in the range \(\epsilon < \gamma < \pi - \epsilon\), where \(0 < \epsilon < \pi/2\). After some simplifications, we find
\[ P_n^2(x) + P_{n+1}^2(x) - 2xP_n(x)P_{n+1}(x) \]
\[ = \frac{2}{\pi n \sin \gamma} \left\{ \cos^2 \left[ \left( n + \frac{1}{2} \right) \gamma - \frac{\pi}{4} \right] + \cos^2 \left[ \left( n + \frac{3}{2} \right) \gamma - \frac{\pi}{4} \right] \right\} \]
\[ + 2 \cos \gamma \cos \left[ \left( n + \frac{1}{2} \right) \gamma - \frac{\pi}{4} \right] \cos \left[ \left( n + \frac{3}{2} \right) \gamma - \frac{\pi}{4} \right] \]
\[ + O(n^{-2} \csc^2 \gamma) \]
\[ = \frac{2}{\pi n} (1 - x^2)^{1/2} + O(n^{-2}(1 - x^2)^{-1}). \]

Making use of (3.8), we obtain

\[ \left\{ P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x) \right\} > \frac{2}{\pi} (1 - x^2)^{-1/2} \]

for sufficiently large \( n \) and \( |x| < 1 - n^{-2/3} \log n \). By the first theorem of Stieltjes, [2, p. 197, (8)] \( |P_n(x)| \leq 4n^{-1/2}(1 - x^2)^{-3/4} \) and by (3.6), \( |P''_n(x)| \leq 8n^{1/2}(1 - x^2)^{-5/4} \). Thus

\[ nP_n(x)P_{n+1}(x) = O((1 - x^2)^{-1/2}), \]

\[ P_n(x)P_n'(x) = O((1 - x^2)^{-3/2}) \]

and

\[ P_{n+1}'(x)P_n(x) + P_n'(x)P_{n+1}(x) = O((1 - x^2)^{-3/2}). \]

By putting these estimates in (3.3), (3.4) and (3.5), we get

\[ (1 - x^2)(P_{n+1}'''P_n' - P_n'''P_{n+1}') \]
\[ = n(n + 1)(P_{n+1}'P_n - P_n'P_{n+1}) + O(n(1 - x^2)^{-3/2}), \]

\[ (1 - x^2)(P_{n+1}''P_n - P_n''P_{n+1}) \]
\[ = 2x(P_{n+1}'P_n - P_n'P_{n+1}) + O((1 - x^2)^{-3/2}) \]

and

\[ (1 - x^2)(P_{n+1}'''P_n - P_n'''P_{n+1}) = \left\{ \frac{16x}{1 - x^2} - n - n^2 \right\} (P_{n+1}'P_n - P_n'P_{n+1}) \]
\[ + O(n(1 - x^2)^{-3/2}), \]
where we have written $P_k, P_k', P_k''$ and $P_k'''$ for $P_k(x), P_k'(x), P_k''(x)$ and $P_k'''(x)$, respectively. This abbreviation is also employed below.

By using (3.11), we finally obtain, for $|x| < 1 - n^{-2/3} \log n$, the estimate

$$
\frac{(P_{n+1}' P_n - P_{n+1}' P_n')}{(P_{n+1}' P_n - P_{n+1}' P_n')} = - n^2 (1 - x^2)^{-1} (1 + O(1/n)),
$$

$$
\frac{(P_{n+1}' P_n - P_{n+1}' P_n')}{(P_{n+1}' P_n - P_{n+1}' P_n')} = n^2 (1 - x^2)^{-1} (1 + O(1/n))
$$

and

$$
\frac{(P_{n+1}' P_n - P_{n+1}' P_n')}{(P_{n+1}' P_n - P_{n+1}' P_n')} = O(n^{2/3} (1 - x^2)^{-1}).
$$

Putting these values for $Q_n(x)/D_n(x), R_n(x)/D_n(x)$ and $S_n(x)/D_n(x)$ in (1.2), the expression enclosed by brackets is estimated by

$$
(3.15) \quad \frac{n^2}{3} \left[ 1 + O\left( \frac{1}{(\log n)^3} \right) \right].
$$

Let $\epsilon = n^{-2/3} \log n$ and $N(c; \epsilon)$ denote the number of zeros of

$$
f(c; x) = c_0 P_0^*(x) + c_1 P_1^*(x) + \cdots + c_n P_n^*(x)
$$

in $-1 + \epsilon \leq x \leq 1 - \epsilon$. By (1.2), we have

$$
\int_{R_n+1} N(c; \epsilon) dP(c) = \frac{1}{\pi} \int_{-1+\epsilon}^{1-\epsilon} \frac{n}{\sqrt{3}} [1 + O((\log n)^{-3})] (1 - x^2)^{-1/2} dx
$$

$$
= \frac{n}{\sqrt{3}} \{ 1 + O((\log n)^{-3}) \}.
$$

To complete the proof of Theorem 1, we observe (cf. [2, p. 250]) that

$$
|P_n(z)| = \frac{1}{\pi} \left| \int_0^\gamma \{ z + i(1 - z^2)^{1/2} \cos \tau \}^n d\tau \right|
$$

and thus for $z = 1 + \epsilon e^{i\theta}$, $|P_n(z)| < (1 + 3\epsilon)^n < 2n^2 \exp(n^{1/3})$. Further $P_n(1) = 1$. We can prove, as in our work [1, p. 722], that

$$
\Pr \left( \max_{0 \leq k \leq n} |c_k| \leq n \right) > 1 - e^{-n^{1/2}},
$$

so that for $z = 1 + \epsilon e^{i\theta}$, we have

$$
(\Pr( |f(c; x)| \geq 4n^4 \exp(n^{1/5})) < \exp(-n^2/3))
$$

and
Let $v(c; \epsilon)$ denote the number of zeros of $f(c; z)$ in $|z - 1| \leq \epsilon$. By making use of Jensen's theorem, we find

$$v(c; \epsilon) \log 2 \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{f(c; 1 + \epsilon e^{i\theta})}{f(c; 1)} \right| d\theta = O(n^{1/3})$$

with probability at least equal with $1 - 2/n$. This shows that the average number of zeros of $f(\cdot; x)$ in $1 - \epsilon \leq x \leq 1$ and similarly in $-1 \leq x \leq -1 + \epsilon$ is $O(n^{1/3})$. Therefore, on using (3.16), we finally obtain the proof of Theorem 1.

**References**


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