PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, January 1971

ON THE UNIVALENCE OF A CERTAIN INTEGRAL

E. P. MERKES AND D. J. WRIGHT

ABSTRACT. We consider the function $g(z)=\int_0^z[f(t)/t]^\alpha dt$ for f in the classes of convex, starlike, and close-to-convex univalent functions, and we determine precisely which values of α yield a close-to-convex function g.

1. **Introduction.** Let S denote the class of functions $f(z)=z+a_2z^2+\cdots$ that are analytic and univalent in the unit disk $E=\{z:|z|<1\}$. Let C, S^*, and K denote respectively the subclasses of S whose members are close-to-convex [2], starlike relative to the origin, and convex in E. For $f\in S$, set

$$g(z)=\int_0^z\left[\frac{f(t)}{t}\right]^\alpha dt, \tag{1}$$

where α is real. Causey [1] has proved that if f is close-to-convex relative to a function $\phi\in K$, and $0\leqq\alpha\leqq 1$, then g is close-to-convex relative to a function in K. Nunokowa [4] recently showed that $g\in S$ provided $f\in S^*$, $0\leqq\alpha\leqq 3/2$, or $f\in K$, $0\leqq\alpha\leqq 3$. In this paper, the following sharp theorems are proved.

THEOREM 1. *If $f\in S^*$, then the function g is in C provided $-1/2\leqq\alpha\leqq 3/2$. If $\alpha\notin[-1/2,3/2]$, there is a function $f\in S^*$ such that the corresponding g is not in S.*

THEOREM 2. *If $f\in K$, then g is in C provided $-1\leqq\alpha\leqq 3$. For $\alpha\notin[-1,3]$, there is a function $f\in K$ such that the corresponding function g is not in S.*

THEOREM 3. *If $f\in C$, then g is in C provided $-1/2\leqq\alpha\leqq 1$. If $\alpha\notin[-1/2,1]$, there is an $f\in C$ such that the corresponding g is not in C.*

2. **Proofs of Theorems 1 and 2.** Kaplan [2] has shown that $g\in C$ if and only if

$$\int_{\theta_1}^{\theta_2}\operatorname{Re}\left[1+re^{i\theta}\frac{g''(re^{i\theta})}{g'(re^{i\theta})}\right]d\theta>-\pi \tag{2}$$

whenever $0\leqq r<1$, $0\leqq\theta_1<\theta_2\leqq 2\pi$. For $f\in S^*$, we have $\operatorname{Re}\{zf'/f\}>0$ for $z\in E$. By (1) it follows that for $\alpha>0$

Received by the editors March 9, 1970.
AMS 1969 *subject classifications*. Primary 3042.
Key words and phrases. Univalent function, starlike, convex, close-to-convex.

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1 + re^{i\theta}\,\frac{g''(re^{i\theta})}{g'(re^{i\theta})}\right] d\theta = \alpha \int_{\theta_1}^{\theta_2} \operatorname{Re}\left[re^{i\theta}\,\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right] d\theta$$
$$+ (1-\alpha)(\theta_2-\theta_1) > (1-\alpha)(\theta_2-\theta_1).$$

The last quantity is not less than $-\pi$ provided $0 \leqq \alpha \leqq 3/2$, and hence $g \in C$. In fact, for $0 \leqq \alpha \leqq 1$,

$$\text{(3)} \qquad \operatorname{Re}\left[1 + \frac{zg''(z)}{g'(z)}\right] = \alpha \operatorname{Re} \frac{zf'(z)}{f(z)} + 1 - \alpha \geqq 1 - \alpha \geqq 0,$$

which implies that $g \in K \subset C$. Finally, when $-1/2 \leqq \alpha \leqq 0$, a result of Marx [**3**] gives $\operatorname{Re}[f(z)/z]^\alpha > 0$. Hence, $\operatorname{Re} g'(z) = \operatorname{Re}[f(z)/z]^\alpha > 0$, and this implies $g \in C$ [**2**].

For $\alpha \notin [-1/2, 3/2]$, set $f(z) = z/(1-z)^2$. Then $g'(z) = (1-z)^{-2\alpha}$, and thus $g(z)$ is univalent in E if and only if $h(z) = (1-z)^{1-2\alpha}$ is univalent in E. By a lemma due to Royster [**5**], the latter is the case if and only if $-1/2 \leqq \alpha \leqq 3/2$, $\alpha \neq 1/2$. When $\alpha = 1/2$, $g(z) = -\ln(1-z)$ which is univalent in E. This completes the proof of Theorem 1.

The proof of Theorem 2 parallels that of the first theorem. If $f \in K$, we utilize the facts that $\operatorname{Re}\{zf'/f\} \geqq 1/2$ and $\operatorname{Re}\{z/f\} > 0$ for $z \in E$ [**3**]. Sharpness follows by consideration of $f(z) = z/(1-z)$.

3. Proof of Theorem 3. The following lemma generalizes a result of Sakaguchi [**6**].

LEMMA. *Let $f(z) = \sum_{n=1}^\infty a_n z^n$, $g(z) = \sum_{n=1}^\infty b_n z^n$ be analytic in E and let $g(z)$ be univalent and starlike relative to the origin in E. If H denotes the convex hull of the image of E under the mapping f'/g', then $f(z)/g(z) \in H$ for $z \in E$.*

PROOF. Let $\psi(w)$ denote the inverse function of $g(z)$ and let $h(w) = f(\psi(w))$. Then

$$\frac{f(z)}{g(z)} = \frac{h(w)}{w} = \frac{1}{w}\int_0^w h'(t)\,dt$$
$$= \frac{1}{w}\int_0^w \frac{f'(\psi(t))}{g'(\psi(t))}\,dt = \frac{1}{\rho}\int_0^\rho \frac{f'(\psi(re^{i\phi}))}{g'(\psi(re^{i\phi}))}\,dr,$$

where $w = \rho e^{i\phi}$, and the result follows.

Suppose $f \in C$. Then there exists a convex function $\phi(z)$, $\phi(0) = 0$, in E such that $\operatorname{Re}\{f'/\phi'\} > 0$ for $z \in E$ [**2**]. If $0 \leqq \alpha \leqq 1$, set

$$\Phi(z) = \int_0^z \left[\frac{\phi(t)}{t}\right]^\alpha dt.$$

Then as in (3) Φ is convex.

Now,

$$\operatorname{Re}\left[\frac{g'(z)}{\Phi'(z)}\right] = \operatorname{Re}\left[\frac{f(z)}{\phi(z)}\right]^{\alpha} > 0$$

by (1) and the lemma. This proves $g \in C$ in this case.

When $-1/2 \leqq \alpha < 0$, set

$$\Phi(z) = e^{-i(1+\alpha)\beta} \int_0^z \left[\frac{\phi(t)}{t}\right]^{1+2\alpha} dt$$

where $\beta = \arg \phi'(0)$. As in the previous case, it follows that $\Phi(z)$ is convex in E. Now

$$\operatorname{Re}\left[\frac{g'(z)}{\Phi'(z)}\right] = \operatorname{Re}\left\{\left[\frac{f(z)}{\phi(z)}\right]^{\alpha}\left[e^{i\beta}\frac{z}{\phi(z)}\right]^{1+\alpha}\right\}. \tag{4}$$

Since $\operatorname{Re}\{z/h(z)\} > 0$ when $h(z) \in K$ [3], we conclude by the lemma that (4) is positive and, hence, that $g \in C$ for $-1/2 \leqq \alpha < 0$.

When $\alpha < -1/2$, the sharpness is a consequence of Theorem 1. Suppose $\alpha > 1$. The function

$$f(z) = \frac{z(1+\mu z)}{(1+z)^2}, \qquad \mu = (\cos\gamma)e^{i\gamma}, \quad 0 < \gamma < \pi,$$

is close-to-convex with respect to $\phi(z) = -ie^{i\gamma}z/(1+z)$ and maps E onto the plane minus the slit

$$w = (1 + i\cot\gamma)/4 + t[1/2 - (1 + i\cot\gamma)/4], \qquad 0 \leqq t < \infty,$$

with $f(e^{i(\pi-2\gamma)}) = (1 + i\cot\gamma)/4$. If $\theta_1 = \pi - 2\gamma$ and $\theta_1 < \theta_2 < \pi$, then as $r \to 1$ and $\theta_2 \to \pi$, we have

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \to \gamma - \pi/2 - \arctan(\cot\gamma).$$

This in turn approaches $-\pi$ as $\gamma \to 0^+$. Thus, for $\alpha > 1$, we have

$$\begin{aligned}
&\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + re^{i\theta}\frac{g''(re^{i\theta})}{g'(re^{i\theta})}\right\} d\theta \\
&\qquad = (1-\alpha)(\theta_2 - \theta_1) + \alpha \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\theta \\
&\qquad < \alpha\{\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})\}.
\end{aligned}$$

The last quantity can be made arbitrarily close to $-\alpha\pi$ by choosing r, θ_2, γ near $1, \pi, 0$ respectively. By (2) it follows that $g \notin C$.

4. A related problem. Let $f\in S$ and set $G(z)=\int_0^z [f'(t)]^\alpha dt$. If $f\in K$, then $zf'(z)$ is in S^*. Hence, by Theorem 1, $G\in C$ whenever $f\in K$ if and only if $-1/2\leqq\alpha\leqq 3/2$. The following result extends a theorem due to Royster [5].

THEOREM 4. *If $f\in C$, then $G\in C$ provided $-1/3\leqq\alpha\leqq 1$. If $\alpha\notin[-1/3, 1]$, there is a function $f\in C$ such that the corresponding $G\notin S$.*

PROOF. The result in case $0\leqq\alpha\leqq 1$ was proved by Royster [5]. Suppose $-1/3\leqq\alpha<0$. If $f\in C$, there is a convex function ϕ in E such that $\phi(0)=0$ and $\operatorname{Re}\{f'(z)/\phi'(z)\}>0$ for $z\in E$. Set

$$\Phi(z) = e^{-i\beta(1+2\alpha)}\int_0^z \left[\frac{\phi(t)}{t}\right]^{1+3\alpha} dt,$$

where $\beta=\arg\phi'(0)$. Then, as in (3), $\Phi(z)$ is convex in E and

$$\frac{G'(z)}{\Phi'(z)} = \left[\frac{f'(z)}{\phi'(z)}\right]^\alpha \left[\frac{z\phi'(z)}{\phi(z)}\right]^\alpha \left[e^{i\beta}\frac{z}{\phi(z)}\right]^{1+2\alpha}.$$

Since $\operatorname{Re}\{z\phi'/\phi\}>0$, $\operatorname{Re}\{e^{i\beta}z/\phi(z)\}>0$, it follows that $\operatorname{Re}\{G'(z)/\phi'(z)\}>0$ for $z\in E$ and, hence, that $G\in C$.

In order to prove the result is sharp, let $f(z)=z(1-z/2)/(1-z)^2$, which is close-to-convex relative to $z/(1-z)$. Then, $G(z)=-\log(1-z)$ if $\alpha=1/3$ and

$$G(z) = \frac{1}{1-3\alpha}\left[(1-z)^{1-3\alpha}-1\right]$$

if $\alpha\neq 1/3$. By a lemma of Royster [5], $G\notin S$ if $\alpha\notin[-1/3, 1]$.

REFERENCES

1. W. M. Causey, *The close-to-convexity and univalence of an integral*, Math. Z. 99 (1967), 207–212. MR **35** #6807.

2. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185. MR **14**, 966.

3. A. Marx, *Untersuchungen über schlichte Abbildungen*, Math. Ann. **107** (1932), 40–67.

4. M. Nunokowa, *On the univalence of a certain integral*, Trans. Amer. Math. Soc. **146** (1969), 439–446.

5. W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. **12** (1965), 385–387. MR **32** #1342.

6. K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75. MR **21** #5734.

UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221