AN EXACT SOLUTION OF THE NONLINEAR DIFFERENTIAL EQUATION

\[ \dot{y} + p(t)y = q_m(t)/y^{2m-1} \]

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Abstract. An exact solution of the nonlinear differential equation

\[ \dot{y} + p(t)y = q_m(t)/y^{2m-1} \]

is found to be

\[ y = \left[ u^m + c(m-1)^{-1}W^{-2} \right]^{1/m} \]

if \( q_m(t) = c(uv)^{m-2} \), \( u \) and \( v \) are independent solutions of \( \dot{y} + p(t)y = 0 \) and \( W \) is their Wronskian.

E. Pinney [1] has shown that the nonlinear differential equation

(1)

\[ \dot{y} + p(t)y = c/y^2 \]

has exact solutions of the form

(2)

\[ y = \left[ u^2 + cW^{-2} \right]^{1/2} \]

when \( y(t_0) = y_0 \neq 0 \) and \( \dot{y}(t_0) = \dot{y}_0 \), for \( c \) an arbitrary constant and \( p(t) \) given. The functions \( u \) and \( v \) are independent solutions of the linear equations

(3)

\[ \dot{y} + p(t)y = 0 \]

for which \( u(t_0) = y_0, \; \dot{u}(t_0) = \dot{y}_0, \; v(t_0) = 0, \; \dot{v}(t_0) \neq 0 \), where their Wronskian

\[ W = uv - vu = \text{const} \neq 0. \]

With \( m \) real and finite, and \( m \neq 0, 1 \), it is possible to show that

(4)

\[ y = \left[ u^m + c(m-1)^{-1}W^{-2} \right]^{1/m} \]

is an exact solution of

(5)

\[ \dot{y} + p(t)y = q_m(t)/y^{2m-1} \]

provided that \( u \) and \( v \) remain independent solutions of (3) and subject to the same conditions above, except that \( v_0 \) need not be zero, and provided that

(6)

\[ q_m(t) = c(uv)^{m-2}. \]

The proof is simple and will be omitted.

Although \( q_m(t) \) clearly restricts the general class of nonlinear equations implied by (5), important physical problems occur with

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initial conditions such as to make the solution (4) physically interesting. Moreover, it is interesting to regard the use of $uv$ in $q_m(t)$ as a method for generating nonlinear differential equations, for which an exact solution is (4). The arbitrary choice of $p(t)$ allows a wide range of possibilities. Taking $p(t) = \pm \omega^2 = \text{const}$ provides two immediate examples:

\begin{align*}
(7) & \quad \ddot{y} + \omega^2 y = c_1 \sin \omega t \cos \omega t y^{m-2}/y^{2m-1}, \\
(8) & \quad \ddot{y} - \omega^2 y = c_1 / y^{2m-1},
\end{align*}

having solutions

\begin{align*}
(9) & \quad y = \left[ a^m \cos^m \omega t + c_m b^m \sin^m \omega t \right]^{1/m}, \\
(10) & \quad y = \left[ a^m e^{m\omega t} + c_m' b^m e^{-m\omega t} \right]^{1/m},
\end{align*}

respectively, where

$$c_m = c_1 / [\omega^2(ab)^m(m - 1)] \quad \text{and} \quad c_m' = c_1 / [4\omega^2(ab)^m(m - 1)].$$

Constants $a$ and $b$ are determined by the initial conditions, while $c_1$, equal to $c(ab)^{m-2}$, is essentially arbitrary.

References


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