SETS OF LATTICE POINTS WHICH CONTAIN A MAXIMAL NUMBER OF EDGES

G. F. CLEMENTS

ABSTRACT. How should one select an \( l \)-element subset of a rectangular array of lattice points (points with integral coordinates) in \( n \)-dimensional Euclidean space so as to include the largest possible number of edges (pairs of points differing in exactly one coordinate)? It is shown that the generalized Macaulay theorem due to the author and B. Lindström contains the (known) solution.

1. Introduction and statement of results. Let \( n \geq 1, k_1 \leq k_2 \leq \cdots \leq k_n \) and \( l \leq (k_1 + 1)(k_2 + 1) \cdots (k_n + 1) = \theta \) be fixed positive integers. \( F_n \) denotes the \( \theta \) \( n \)-tuples \( x = (x_1, x_2, \cdots, x_n) \) of integers \( x_i, 0 \leq x_i \leq k_i \), \( i = 1, 2, \cdots, n \), ordered lexicographically—i.e. \( x < y \) if \( x_i < y_i \) for the smallest integer \( i \) such that \( x_i \neq y_i \). It will be helpful to imagine the elements of \( F_n \) arrayed in a matrix of \( k_1 + 1 \) rows and \( \theta/(k_1 + 1) \) columns by writing them in increasing order from left to right and top to bottom.

Let \( A_l \) denote an \( l \)-element subset of \( F_n \). An edge is an unordered pair \((x, y)\) of \( n \)-tuples which disagree at exactly one place. A subset \( A \) of \( F_n \) contains the edge \((x, y)\) if and only if \( x \in A \) and \( y \in A \). \( E(A) \) denotes the number of edges \( A \) contains.

We now state two theorems. Theorem 1 is contained in Lindsey's paper [7] while Theorem 2 is Corollary 3 of the generalized Macaulay theorem [2]. The content of this paper is that these two theorems are equivalent.

**Theorem 1.** \( \max E(A_l) = E(S_l) \) where the maximum is taken over all \( l \)-element subsets of \( F_n \) and \( S_l \) denotes the first \( l \) elements of \( F_n \).

The sets \( A_l \) for which the maximum is attained are also characterized in Lindsey's paper.

In order to state the second theorem, we define the set-valued function \( \Gamma \) on \( F_n \) by \( \Gamma(\emptyset) = \emptyset \),
\[ \Gamma(x) = \{(x_1 - 1, x_2, \ldots, x_n), (x_1, x_2 - 1, x_3, \ldots, x_n), \ldots, (x_1, x_2, \ldots, x_n - 1)\} \cap F_n \]

and call a subset \( H \) of \( F_n \) closed if and only if \( \Gamma(H) \subseteq H \), where \( \Gamma(H) = \bigcup_{x \in H} \Gamma(x) \). Notice that \( S_i \) is closed. Finally define

\[ \alpha(x) = x_1 + x_2 + \cdots + x_n \quad \text{and} \quad \alpha(H) = \sum_{x \in H} \alpha(x). \]

**Theorem 2.** \( \max \alpha(H_i) = \alpha(S_i) \) where the maximum is taken over all closed \( l \)-element subsets of \( F_n \).

It is not difficult to verify that \( \alpha \) and \( E \) agree on closed sets; hence \( \alpha(S_i) \) can be replaced by \( E(S_i) \) in the statement of Theorem 2. If this is done, the similarity between the two theorems becomes even greater. This similarity is noticed implicitly in the paper [6] of J. B. Kruskal. More precisely, Kruskal points out that the \( k_1 = k_2 = \cdots = k_n = 1 \) case of Theorem 1, which is contained in the papers of Harper [3] and Bernstein [1], is analogous to a result of his [5]. (Kruskal's result has been rediscovered by G. Katona and applied to a problem concerning the existence of certain subsets of a finite set [4].) Actually it can be shown that the \( k_1 = k_2 = \cdots = k_n = 1 \) case of Theorem 2 follows from Kruskal's result and that Kruskal's result contains the \( k_1 = k_2 = \cdots = k_n = 1 \) special case of the generalized Macaulay theorem.

2. **The equivalence of Theorems 1 and 2.** It is clear that Theorem 1 implies Theorem 2 since if one takes the maximum only over closed sets he has

\[ \alpha(S_i) \leq \max \alpha(H_i) = \max E(H_i) \leq E(S_i) = \alpha(S_i). \]

Conversely, assume Theorem 2 and suppose that \( \overline{A}_i \) is maximal: \( E(\overline{A}_i) = \max E(A_i) \). We show that \( \overline{A}_i \) can be replaced by \( S_i \) without decreasing the number of edges. This is obvious for 1-tuples. Assuming it is true for \( t \)-tuples, \( t = 1, 2, \cdots, (n - 1) \), we consider \( n \)-tuples. For a subset \( G \) of \( F_n \), let \( G_i \) denote the elements of \( G \) which begin with \( i \), \( i = 0, 1, \cdots, k_i \); thus the elements of \( G_i \) appear in the \( i \)-th row of \( F_n \). Let \( a_i \) denote the number of elements in \( (\overline{A}_i)_i \), \( i = 0, 1, \cdots, k_i \). One easily convinces himself that it is no loss of generality to assume that \( a_0 \geq a_1 \geq \cdots \geq a_{k_i} \) since \( \overline{A}_i \) could be replaced by a set having \( E(\overline{A}_i) \) edges for which this is true. We will say that an edge \( (x, y) \) in \( \overline{A}_i \) is an \((i, j)\) edge if \( i \leq j \), if \( x \in (\overline{A}_i)_i \), and \( y \in (\overline{A}_i)_j \). If \( N_{(i, j)}(\overline{A}_i) \) is the number of \((i, j)\) edges in \( (\overline{A}_i)_i \), then
If $A_1$ is replaced by the set $A_1'$ consisting of the first $a_i$ elements of $(F_n)_i$, $i = 0, 1, \cdots, k_1$, no summand is decreased. That $N(i,0) (A_1') \geq N(i,0) (A_1)$ follows from the $k_2, k_3, \cdots, k_n$ case of the induction hypothesis; that $N(i,j) (A_1') \geq N(i,j) (A_1)$ if $i < j$ follows from the fact that $N(i,j) (A_1') \leq a_j$ (since $(A_1)_j$ has $a_j$ elements) while $N(i,j) (A_1') = a_j$ (since the $p$th elements in $(A_1')_i$ and $(A_1')_j$ constitute an edge, $p = 1, 2, \cdots, a_j$; we are using here that $a_i \geq a_j$). Also $A_1'$ is closed since if $x$ is the $p$th element in $(A_1')_i$, $1 \leq p \leq a_i$, then each element $z \in \Gamma(x)$ is either a smaller element in $(F_n)_i$ and therefore in $(A_1')_i$, since $(A_1')_i$ is the first $a_i$ elements of $(F_n)_i$, or $z$ is the $p$th element of $(F_n)_i$, $1 \leq p \leq a_i$, of $(A_1')_i$ since $a_i \geq a_i$.

Thus $A_1$ has been replaced by a closed set $A_1'$ having at least as many edges. If we now replace $A_1'$ by $S_1$ we again do not decrease the number of edges in view of Theorem 2 and the fact that $\alpha$ and $E$ agree on closed sets. This completes the induction. The equality $E(A_1) = E(S_1)$ now follows from

$$\alpha(S_1) = E(S_1) \leq \max E(A_1) = E(A_1) \leq E(S_1) = \alpha(S_1).$$

**References**


**University of Colorado, Boulder, Colorado 80302**