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ON PERFECT GROUP RINGS

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ABSTRACT. It is shown that the group ring AG of the group G over the ring A is perfect if and only if A is perfect and G is finite. (Perfect rings were characterized by H. Bass in 1960.)

1. **Introduction.** All rings are assumed to have a unit element. A ring R is called semiprime if its prime radical $\operatorname{rad}(R)$ is 0, and semiprimitive if its Jacobson radical $\operatorname{Rad}(R)$ is 0. The ring R is perfect if $R/\operatorname{Rad}(R)$ is artinian and $\operatorname{Rad}(R)$ is left T-nilpotent (i.e. for every sequence $\{a_i\}$ in $\operatorname{Rad}(R)$ there exists an n such that $a_1a_2 \cdots a_n=0$). Equivalently, R is perfect if it satisfies the descending chain condition on principal right ideals. If R is a perfect ring then so is every homomorphic image of R, and so is $R_{(n)}$, the ring of all n by n matrices over R. These results are due to H. Bass [1].

If A is a ring and G is a group, AG will denote the group ring of G over A. If H is a subgroup of G then ωH will denote the right ideal of AG generated by $\{1-h\colon h\in H\}$; if H is normal then ωH is an ideal and $AG/\omega H\cong A(G/H)$. The fundamental ideal ωG of AG will be denoted Δ; $A\cong AG/\Delta$. If I is a right ideal of AG then IG will denote the right ideal of AG generated by the subset I; if I is an ideal then IG is an ideal and $AG/IG\cong(A/I)G$. The group ring AG is (von Neumann) regular if and only if A is regular, G is locally finite, and the order of every finite subgroup of G is a unit in A; AG is artinian if and only if A is artinian and G is finite; and AG is semiprime if and only if A is semiprime and G has no finite normal subgroups whose orders are zero-divisors in A. These results may be found in Connell's paper [3] and his appendix to Lambek's book [4, Appendix 3].

This work forms part of the author's Ph.D. thesis at McGill University. Conversations with Professors J. Lambek and I. Connell were most helpful.

2. **Sufficiency.** In this section we assume that A is perfect and G is finite and show that $\operatorname{Rad}(AG)$ is left T-nilpotent and $AG/\operatorname{Rad}(AG)$ is artinian.

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LEMMA 1. *If G is finite of order n then there is a ring embedding of AG into $A_{(n)}$ which sends* $\mathrm{Rad}(A)G$ *into* $\mathrm{Rad}(A_{(n)})$.

PROOF. Since $AG \cong A^n$ as right A-modules, the endomorphism ring $\mathrm{End}_A(AG) \cong A_{(n)}$. Left multiplication by an element of AG is a right A-homorphism of AG into itself and this correspondence is clearly an embedding of the ring AG into the ring $\mathrm{End}_A(AG)$.

Since elements of A commute with elements of G, an element a of A is mapped onto the matrix with a's on the diagonal and 0's elsewhere. Thus $\mathrm{Rad}(A)$ is mapped into $\mathrm{Rad}(A)_{(n)} = \mathrm{Rad}(A_{(n)})$, an ideal. The result follows.

PROPOSITION 1. *If A is perfect and G is finite then AG is perfect.*

PROOF. Let $\bar{A} = A/\mathrm{Rad}(A)$. Then $\bar{A}G$ is artinian. By [3, Proposition 9], $\mathrm{Rad}(A)G \subseteq \mathrm{Rad}(AG)$. Then $\bar{A}G \cong AG/\mathrm{Rad}(A)G$ maps onto $AG/\mathrm{Rad}(AG)$ and $AG/\mathrm{Rad}(AG)$ is artinian.

The canonical epimorphism of AG onto $\bar{A}G$ takes $\mathrm{Rad}(AG)$ into $\mathrm{Rad}(\bar{A}G)$, this is, $\mathrm{Rad}(AG)/\mathrm{Rad}(A)G \subseteq \mathrm{Rad}(\bar{A}G)$. Since $\bar{A}G$ is artinian, $\mathrm{Rad}(\bar{A}G)$ is nilpotent. But $\mathrm{Rad}(A)G \subseteq \mathrm{Rad}(A_{(n)})$ which is left T-nilpotent since $A_{(n)}$ is perfect. Thus $\mathrm{Rad}(AG)$ is left T-nilpotent.

3. Necessity when G is abelian.

LEMMA 2. *If AG is perfect then G is a torsion group.*

PROOF. If $g \in G$ does not have finite order then the cyclic subgroups generated by g^{2^n} for $n \geq 0$ form an infinite descending chain. Applying ω yields an infinite descending chain of right ideals of AG, which are principal by [3, Proposition 1].

PROPOSITION 2. *If AG is perfect then so is A. If in addition, G is abelian, then G is finite.*

PROOF. If AG is perfect then so is A since $A \cong AG/\Delta$.

To show that G is finite we may assume without loss of generality that $A = E_{(n)}$ where E is a skewfield, since $A/\mathrm{Rad}(A)$ is a direct sum of rings of this type. Since G is an abelian torsion group, G may be written as $G_p \times H$, where p is the characteristic of E, G_p is a p-group, and the order of every element of H is prime to p. (If E has characteristic 0 take G_p to be trivial and $H = G$.) Burgess [2] has shown that H must be finite.

Suppose that G_p is infinite. Then $AG_p \cong AG/\omega H$ is perfect. If $g \in G_p$ then $(1-g)^{p^n} = 0$ where p^n is the order of g. Since $1-g$ is in the centre, $1-g \in \mathrm{Rad}(AG_p)$. Construct a sequence $\{g_i\}$ in G_p so that $g_1 \neq 1$ and g_n is not in the (finite) subgroup generated by $\{g_1, \cdots, g_{n-1}\}$. The product $\prod_{i=1}^n (1-g_i)$ is never 0, since the term $\prod_{i=1}^n g_i$ does not cancel. This contradicts the T-nilpotence of $\mathrm{Rad}(AG_p)$.

4. **Reduction to the abelian case.** In this section it is shown that if AG is perfect and G is infinite then G has an infinite abelian subgroup H and AH is perfect, a contradiction. Without loss of generality, we continue our assumption that $A = E_{(n)}$, where E is a skewfield.

LEMMA 3. *If AG is perfect and H is a subgroup of G then AH is perfect.*

PROOF. $AG = \bigoplus_i AHg_i$ where the g_i run over a set of coset representatives for G/H. If I is a principal right ideal of AH then $IG = \bigoplus_i Ig_i$ is a principal right ideal of AG. Thus a descending chain of principal right ideals in AH gives rise to a similar chain in AG.

LEMMA 4. *If I is a left T-nilpotent ideal of a ring R then $I \subseteq \mathrm{rad}(R)$. Hence if R is perfect, then* $\mathrm{Rad}(R) = \mathrm{rad}(R)$.

PROOF. The prime radical $\mathrm{rad}(R)$ is the set of all strongly nilpotent elements of R [4, p. 56]. Clearly an element of a T-nilpotent ideal is strongly nilpotent.

LEMMA 5. *A group G which has infinitely many finite normal subgroups has an infinite abelian subgroup.*

PROOF. Without loss of generality we may assume that G is the union of a countable chain of finite normal subgroups H_i. Since an infinite set of commuting elements generates an infinite abelian subgroup, if G does not contain an infinite abelian subgroup then there exists a finite set $S = \{g_1, \cdots, g_m\}$ of commuting elements which cannot be enlarged. Since $G = \bigcup_{i=1}^\infty H_i$, $S \subseteq \bigcup_{i=1}^n H_i = H_n$ for some n. Since H_n is finite, the index of its centraliser C in G is finite. Since G is infinite, C is infinite and so there exists $g \in C$ such that $g \notin S$. Since g commutes with every element of S, g may be added to S and we have reached a contradiction.

PROPOSITION 3. *If AG is perfect then either G is finite or G has an infinite abelian subgroup.*

PROOF. Let $A = E_{(n)}$. If E has characteristic 0 then AG is semiprime, hence semiprimitive by Lemma 4. Thus $AG \cong AG/\mathrm{Rad}(AG)$ is artinian and G is finite.

If E has characteristic $p > 0$ let S be the set of natural numbers $\{n: G$ has a normal subgroup of order $p^n m$ for some $m\}$. If S is finite let n be maximal and let H_n be a normal subgroup whose order is divisible by p^n. By the maximality of n, G/H_n has no finite normal subgroup whose order is divisible by p. Therefore $A(G/H_n)$ is semiprime. Since $A(G/H_n)$ is perfect, G/H_n is finite. Since H_n is finite, so is G.

If S is infinite then G has infinitely many finite normal subgroups. By Lemma 5, G contains an infinite abelian subgroup.

This completes the proof of the theorem.

THEOREM. *The group ring AG is perfect if and only if A is perfect and G is finite.*

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