L² ASYMPTOTES FOR THE KLEIN-GORDON EQUATION

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Abstract. An approximation \( a(x,t) \) is obtained for solutions \( u(x,t) \) of the Klein-Gordon equation. \( a(x,t) \) can be expressed in terms of the Fourier transforms of the Cauchy data and it is shown that \( \| a(\cdot,t) - u(\cdot,t) \| \to 0 \) as \( t \to \infty \). This result is applied to show how energy distributes among various conical regions.

A wide class of solutions to the Klein-Gordon equation

\[
\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial^2 u}{\partial t^2} = u
\]

can be written in the form

\[
\begin{align*}
   u(x,t) &= (2\pi)^{-n/2} \int e^{i\xi \cdot x} \left[ F(y) \cos t\sqrt{1 + y^2} + G(y) \sin t\sqrt{1 + y^2} \right] d^n y \\
   &= (2\pi)^{-n/2} \int e^{i\xi \cdot x} \left[ \psi(y) \exp[it\sqrt{1 + y^2}] + \phi(y) \exp[-it\sqrt{1 + y^2}] \right] d^n y
\end{align*}
\]

where \( F = \psi + \phi, G = i(\psi - \phi) \) are in \( L^2(\mathbb{R}^n) \) and the integral over \( \mathbb{R}^n \) is interpreted in the sense of Plancherel's theorem. Our main result is

**Theorem 1.** Define \( a(x,t) = 0 \) for \( |x| > t \) and for \( |x| < t \) define

\[
a(x,t) = \left\{ e^{i\phi(x,t)} \psi \left( \frac{-x}{\sqrt{t^2 - x^2}} \right) + e^{-i\phi(x,t)} \phi \left( \frac{x}{\sqrt{t^2 - x^2}} \right) \right\} \rho(x,t),
\]

where \( \psi \) and \( \phi \) are the same \( L^2 \) functions as in (1). Then

\[
\| u(\cdot,t) - a(\cdot,t) \|^2 \equiv \int | u(x,t) - a(x,t) |^2 d^n x \to 0 \quad \text{as } t \to \infty.
\]

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Before starting on the proof of Theorem 1 we mention a corollary. Define
\[
\chi(x, t) = \begin{cases} 
1 & \text{if } |x| < t, \\
0 & \text{otherwise.}
\end{cases}
\]

**Corollary 1.**
\[
\lim_{t \to \infty} \|\chi u(\cdot, t)\|^2_2 = \|\psi\|^2_2 + \|\phi\|^2_2 = (\|F\|^2_2 + \|G\|^2_2)/2.
\]

**Proof.** By Theorem 1, \(\lim_{t \to \infty} \|(1 - \chi)u(\cdot, t)\|^2_2 = 0\) and hence
\[
\lim_{t \to \infty} \|u(\cdot, t)\|^2_2 - \|\chi u(\cdot, t)\|^2_2 = 0.
\]
Thus the corollary follows from the fact (see Brodsky [1]) that
\[
\lim_{t \to \infty} \|\chi u(\cdot, t)\|^2_2 = (\|F\|^2_2 + \|G\|^2_2)/2.
\]

**Remark.** Let \(V_n\) denote the volume of the unit ball in \(R^n\) so that
\[
\|\chi u(\cdot, t)\|^2 \leq V_n t^n \|\chi u(\cdot, t)\|^2_2.
\]
Applying the corollary one sees that if \(\|\chi u(\cdot, t)\|_\infty = O(t^{-n/2})\) as \(t \to \infty\) then \(u = 0\), a special case of a result by Littman [3].

The above corollary can be extended to the case where \(\chi\) is replaced by the characteristic function of other cones (see Corollary 1'). Theorem 1 will be deduced from Theorem 1' below. The proof of Theorem 1' is based on

**Lemma 1.** Define
\[
W_\lambda(x) = (2\pi)^{-n/2} \int e^{ix \cdot \eta} \exp\left[-i\sqrt{1 + \eta^2}\right](1 + \eta^2)^{-n/4} d\eta.
\]
Then, for every \(f \in L^1(R^n)\) and \(\lambda \in R^n\),
\[
\lim_{t \to \infty} t^{n/2} e^{-in\lambda/4} \exp[i\lambda/\sqrt{1 + \lambda^2}] W_\lambda * f\left(\frac{\lambda t}{\sqrt{(1 + \lambda^2)}}\right) = \hat{f}(\lambda).
\]

**Proof.** In [5] it is shown that for \(t > 0\),
\[
W_\lambda(x) = t^{-n/2} e^{-in\lambda/4} \exp[-\sqrt{(x^2 - t^2)}] + R_t(x)
\]
where \(\|R_t\|_\infty = O(t^{-n/2})\) as \(t \to \infty\). By \(\sqrt{(x^2 - t^2)}\) we mean the value that lies on the positive imaginary axis when \(|x| < t\) and on the positive real axis when \(|x| > t\). Since \(\|R_t\|_\infty = O(t^{-n/2})\) it is clear that \(t^{n/2} \|R_t * f\|_\infty = O(t^{-1})\) and hence
\[
\lim_{t \to -\infty} t^{n/2}e^{i\pi t/4} \exp[i t \sqrt{(1 + \lambda^2)]} R_\epsilon \ast f \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right) = 0.
\]

Thus to complete the proof of the lemma we must show
\[
\lim_{t \to -\infty} (2\pi)^{-n/2} \int \exp \left( \frac{it}{\sqrt{(1 + \lambda^2)}} \right)
- \sqrt{\left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} - x \right)^2 - t^2} \right) f(x) \, d^n x
= (2\pi)^{-n/2} \int e^{-\lambda \cdot x} f(x) \, d^n x = \tilde{f}(\lambda).
\]

But this is a consequence of Lebesgue's dominated convergence theorem because
\[
\lim_{t \to -\infty} \frac{it}{\sqrt{(1 + \lambda^2)}} - \sqrt{\left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} - x \right)^2 - t^2} = -i\lambda \cdot x.
\]

For \( \phi \in L^2(\mathbb{R}^n) \) define
\[
(3) \quad U_\phi(x) = (2\pi)^{-n/2} \int e^{ix \cdot y} \exp[-it \sqrt{(1 + y^2)}] \phi(y) \, d^n y.
\]

**Corollary 2.** If there exists \( f \in L^1(\mathbb{R}^n) \) such that \( \tilde{f}(\lambda) = (1 + \lambda^2)^{(n+2)/4} \phi(y) \) then
(i) \( \| U_\phi \|_\infty \leq (2\pi)^{-n/2} \| f \|_1 \| W_t \|_\infty \sim (2\pi)^{-n/2} \| f \|_1 t^{-n/2} \) as \( t \to \infty \),
(ii) \( \lim_{t \to -\infty} t^{n/2} e^{i\alpha(\lambda, t)} U_\phi \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right) = \tilde{f}(\lambda) = (1 + \lambda^2)^{(n+2)/4} \phi(\lambda), \)

where \( \alpha(\lambda, t) = n\pi/4 + t/\sqrt{(1 + \lambda^2)} \).

**Proof.** Both (i) and (ii) follow easily from the fact that
\[
U_\phi(x) = (2\pi)^{-n/2} \int e^{ix \cdot \lambda} \exp[-it \sqrt{(1 + \lambda^2)}] (1 + \lambda^2)^{-(n+2)/2} \tilde{f}(\lambda) \, d^n \lambda
= (2\pi)^{-n/2} \int W_t(x - y) f(y) \, d^n y = W_t \ast f(x).
\]

**Remark.** The proof of Corollary 2 follows the approach used by Brodsky [2] and Segal [7, pp. 95–98] to obtain bounds like that given by (i). Recently, I became aware of a different approach by Littman [4] which when applied to the present situation yields (i) and (ii) with different assumptions on \( \phi \). Using Littman's approach Theorem 1 can be extended to more general situations (see [6]).
Motivated by (ii) of Corollary 2 we define an approximation $A\phi(x)$ to $U\phi(x)$ by requiring

$$\tag{4} t^{n/2}e^{-ia(0,1)}A\phi\left(\frac{\lambda t}{\sqrt{(1 + \lambda^2)}}\right) = (1 + \lambda^2)^{(n+2)/4}\phi(\lambda), \quad \lambda \in \mathbb{R}^n, \ t > 0.$$ 

To see how this works out consider the transformation

$$T_t: \lambda \rightarrow x = \frac{\lambda t}{\sqrt{(1 + \lambda^2)}}$$

that maps $\mathbb{R}^n$ onto the ball $|x| < t$. Since $x^2 = \lambda^2 t^2/(1 + \lambda^2)$ we have $\lambda^2(t^2 - x^2) = x^2$ and hence $T_t^{-1}: x \rightarrow \lambda = x/\sqrt{(t^2 - x^2)}$. Taking $\lambda = T_t^{-1}(x)$ in (4) and multiplying by $t^{-n/2}e^{-ia}$ gives

$$A\phi(x) = \exp[-ia(T_t^{-1}(x), t)]t^{-n/2}\left(1 + \frac{x^2}{t^2 - x^2}\right)^{(n+2)/4}\phi(T_t^{-1}(x))$$

$$= e^{-\theta(x,t)}\rho(x,t)\phi\left(\frac{x}{\sqrt{(t^2 - x^2)}}\right), \quad |x| < t,$$

where $\theta$ and $\rho$ are the functions defined in Theorem 1.

**Theorem 1'.** For $\phi \in L^2(\mathbb{R}^n)$ define $U\phi$ and $A\phi$ by (3) and (5). Then

(i) $\|A\phi\|_2 = \|\phi\|_2 = \|U\phi\|_2$,

(ii) $\lim_{t \to \infty} ||U\phi - A\phi||_2 = 0$,

(iii) $\lim_{t \to \infty} \int_{|x| > t} |U\phi(x)|^2 dx = 0$,

where

$$\|f\|_2^2 = \left\{ \int_{|x| < t} |f(x)|^2 dx \right\}^{1/2}. $$

**Proof.** It is not difficult to check that the Jacobian of $T_t$ is

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(\lambda_1, \ldots, \lambda_n)} = t^n(1 + \lambda^2)^{-(n+2)/2} = \rho^{-2}(T_t(\lambda), t).$$

Thus by the change of variable theorem for multiple integrals

$$\|f\|_2^2 = \int |f(T_t(\lambda))|^2 \rho^{-2}(T_t(\lambda), t) d^n\lambda.$$

Taking $f = A\phi$ in (6) we obtain the left-hand side of (i). The other half of (i) is Parseval's equality.

To prove (ii) let $\epsilon > 0$ and choose $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $||\phi - \phi||_2 < \epsilon/3$. Applying (i) to $\phi - \bar{\phi}$ we have

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Thus
\[ \| U\phi - A\phi \|_{2,t} \leq \| U\phi - U_0\phi \|_{2,t} + \| U_0\phi - A\phi \|_{2,t} + \| A\phi - A_0\phi \|_{2,t} < \varepsilon/3 + \| U\phi - A\phi \|_{2,t} + \varepsilon/3, \]
so to prove (ii) we need only show there exists \( \tau \) such that \( t > \tau \) implies \( \| U\phi - A\phi \|_{2,t} < \varepsilon/3 \). Since this amounts to proving (ii) with \( \phi \) replaced by \( \tilde{\phi} \), we simply assume \( \phi \in C_c^\infty (\mathbb{R}^n) \).

Applying (6) we have
\[ \| U\phi - A\phi \|_{2,t}^2 = \int \left| U\phi \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right) - A\phi \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right) \right|^2 \cdot \rho^{-2}(T_t(\lambda), t)d^n\lambda \]
(7)
\[ = \int \left| e^{-i\alpha (\lambda_1)}[g_t(\lambda) - \phi(\lambda)] \right|^2 d^n\lambda \]
where
\[ g_t(\lambda) = e^{i\alpha (\lambda_1)}I_n^{1/2}(1 + \lambda^5)(n+2)/4 U\phi \left( \frac{\lambda t}{\sqrt{(1 + \lambda^2)}} \right). \]
Clearly \( \| U\phi \|_{2,t}^2 = \int |g_t(\lambda)|^2 d^n\lambda \) and hence
\[ \| g_t \|_2 \leq \| U\phi \|_2 = \| \phi \|_2. \]
(8)
Since \( \phi \in C_c^\infty (\mathbb{R}^n) \) part (ii) of Corollary 2 can be used to conclude that for every \( \lambda \in \mathbb{R}^n \)
\[ \lim_{t \to \infty} g_t(\lambda) = \phi(\lambda). \]
(9)
An application of Fatou’s lemma and Egoroff’s theorem shows that (8) and (9) imply \( \lim_{t \to 0} \| g_t - \phi \|_2 = 0 \) which in view of (7) establishes (ii).

To prove (iii) we must show \( \lim_{t \to 0} \| U\phi \|_{2,t}^2 - \| U\phi \|_{2,t}^2 = 0 \) which is obvious from (i) and (ii).

**Proof of Theorem 1.** Since
\[ u(x, t) = U\phi(-x) + U\phi(x), \quad a(x, t) = A\phi(-x) + A\phi(x), \]
Theorem 1 follows immediately from Theorem 1'.

Let \( \Gamma \) be a cone inside the cone \( \{(x, t) : |x| < t\} \). That is, assume \( (x, t) \in \Gamma \) implies \( |x| < t \) and \( (sx, st) \in \Gamma \) for all \( s > 0 \). Put
\[
B = \left\{ \frac{x}{\sqrt{1 - x^2}} : (x, 1) \in \Gamma \right\}
\]
and let \(\beta(\lambda)\) be the characteristic function of \(B\). Then the characteristic function \(\gamma(x,t)\) of \(\Gamma\) is zero for \(|x| > t\) and satisfies
\[
\gamma(x,t) = \beta \circ T_t^{-1}(x) = \beta\left(\frac{x}{\sqrt{t^2 - x^2}}\right) \quad \text{for} \quad |x| < t.
\]

**Corollary 1'.** Let \(\gamma(x,t), B\) be as above and let \(u(x,t), \phi(y), \psi(y)\) be as in Theorem 1. Then

\[
\lim_{t \to \infty} \|\gamma u(\cdot, t)\|_2^2 = \int_B |\psi(-\lambda)|^2 + |\phi(\lambda)|^2 d^n\lambda.
\]

**Remark.** Similar expressions can be obtained for the kinetic and potential energies \(\|\gamma u_t\|_2^2, \|\gamma u\|_2^2 + \sum \|\gamma u_{x_i}\|_2^2\) since \(u_t\) and \(u_{x_i}\) can also be written in the form of equation (1). This gives an extension of the Virial theorem stated in Brodsky [1].

**Proof.** By Theorem 1 it suffices to prove the corollary with \(u(x,t)\) replaced by \(a(x,t)\). When \(|x| < t\) we have

\[
\gamma(x,t) a(x,t) = \beta\left(\frac{x}{\sqrt{t^2 - x^2}}\right) \left\{ e^{i\theta(x,t)} \psi\left(\frac{-x}{\sqrt{t^2 - x^2}}\right) + e^{-i\theta(x,t)} \phi\left(\frac{x}{\sqrt{t^2 - x^2}}\right) \right\} \rho(x,t),
\]

and hence, by (6),

\[
\|\gamma a(\cdot, t)\|_{2,t}^2 = \int B(\lambda) \left| e^{ia(\lambda,t)} \psi(-\lambda) + e^{-ia(\lambda,t)} \phi(\lambda) \right|^2 d^n\lambda.
\]

Since \(\left| e^{ia(\psi)} + e^{-ia(\phi)} \right|^2 = \left| \psi \right|^2 + \left| \phi \right|^2 + e^{2ia(\psi)\phi} + e^{-2ia(\psi)\phi} \) the corollary follows from

**Lemma 2.** For \(f \in L^1(R^n)\) define \(I_f(t) = \int \exp[it/\sqrt{(1 + \lambda^2)}] f(\lambda) \ d^n\lambda.\) Then \(\lim_{t \to \infty} I_f(t) = 0.\)

**Proof.** Since \(C_c(R^n)\) is dense in \(L^1(R^n)\) and since \(\left| I_f(t) - I_g(t) \right| \leq \|f - g\|_1\) it suffices to prove the lemma when \(f\) is continuous and has compact support. Switching to spherical coordinates gives

\[
I_f(t) = \int_0^\infty \exp[it/\sqrt{(1 + r^2)}] F(r) dr, \quad F(r) = r^{n-1} \int_{|\psi| = 1} f(ry) dS(y).
\]

Applying the change of variable \(u = 1/\sqrt{(1 + r^2)}\) yields
\[ I_\varepsilon(t) = \int_0^1 e^{i\varepsilon u} \phi(u) \, du, \quad \phi(u) = F\left(\frac{\sqrt{1 - u^2}}{u}\right) \frac{1}{u^2\sqrt{1 - u^2}}. \]

Since \( \sqrt{1-u^2} \phi(u) \in \mathcal{C}_c((0, 1]) \), the lemma follows from the Riemann-Lebesgue theorem.

**Remark on \( L^p \) behavior.** By Hölder's inequality

\[ \| \gamma u(\cdot, t) \|_{L^2} \leq \| \gamma(\cdot, t) \|_2 \| \gamma u(\cdot, t) \|_{L^2} \]

\[ = (\| \gamma(\cdot, t) \|_1)^{1/2} \| \gamma u(\cdot, t) \|_{L^2}^{1/2} \]

\[ = \left( \| \gamma(\cdot, t) \|_{L^1}^{1/2} \right)^{{(1-2/p)}^{1/2}} \| \gamma u(\cdot, t) \|_{L^p} \]

where \( 1 - 1/q = 1/q' = 2/p \). Thus Corollary 1' gives a lower bound for

\[ \lim \inf_{t \to 0} \frac{\varepsilon}{2-n/p} \| \gamma u(\cdot, t) \|_{L^p} \]

which is positive unless \( \phi(\lambda) \) and \( \psi(-\lambda) \) vanish for \( \lambda \in B \).

On the other hand, by using Corollary 2(i) and the estimate

\[ \| f \|_{L^p} \leq \| f \|_{L^2} \| f \|_{L^{2-p}}, \]

\( 2 \leq p < \infty \) one can show

\[ \| u(\cdot, t) \|_{L^p} = O(t^{-n/2}l^{n/p}), \quad \| u(\cdot, t) - a(\cdot, t) \|_{L^p} = o(t^{-n/2}l^{n/p}) \]

as \( t \to \infty \), provided \( \phi \) and \( \psi \) satisfy the condition of Corollary 2.

**References**


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