A PRIMARY DECOMPOSITION FOR TORSION MODULES

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Abstract. A definition of primary module is given and a theorem is proved characterizing rings for which each torsion module, in the sense of S. E. Dickson, decomposes as a direct sum of its primary submodules. This theorem is applied to obtain a generalization of Fuchs' theorem on the additive group structure of Artinian rings.

1. Introduction. S. E. Dickson [2], [3] has investigated a primary decomposition for torsion modules over an arbitrary ring and Alin [1] has characterized rings for which this primary decomposition holds. The purpose of this note is to define "primary module" in such a way that the primary decomposition holds for a larger class of rings, in particular, for all Noetherian rings.

All rings $R$ have a unit and modules are unitary left $R$-modules. If $M$ is an $R$-module, $M^+$ denotes the underlying additive group of $M$. It is well known that if $S$ is a simple $R$-module, then $S^+$ is a direct sum of copies of $Z_p$, the cyclic group of order $p$, or a direct sum of copies of $Q$, the additive group of rational numbers. In the first case we say that $S$ is of type $p$ and in the second, $S$ is of type $Q$.

Let $p_1, p_2, \cdots$ be an indexing of the positive primes and for each $i = 0, 1, \cdots$ let $S_i$ be a representative set of simple $R$-modules of type $p_i$. Let $S_0$ be a representative set of simple $R$-modules of type $Q$. For $i = 0, 1, \cdots$ let $S_i$ be the torsion class generated by $S_i$ and let $S$ be the torsion class generated by $\bigcup_{i=0}^\infty S_i$ [4]. Thus $S_i$ (respectively $S$) is the class of all modules $M$ such that each nonzero homomorphic image of $M$ has a submodule isomorphic to a member of $S_i$ (respectively, $\bigcup_{i=0}^\infty S_i$). The classes $S_i, S_0, S_1, \cdots$ are closed under submodules, direct sums, extensions and homomorphic images. It follows that each module $M$ has a unique largest submodule $M_i$ in $S_i$. If $M \in S_i$, $i \geq 1$, $M$ is $p_i$-primary and if $M \in S_0$, $M$ is $Q$-primary. The modules $M \in S_0$ are called torsion. The primary decomposition holds for a ring $R$ if and only if for each $M \in S_0$, $M = \sum_{i=0}^\infty M_i$ (direct), i.e., each torsion module is a direct sum of its primary submodules.

For an $R$-module $M$, $\text{Soc}(M)$ denotes the socle of $M$. We let $T^1(M) = \text{Soc}(M)$ and extend to an ascending chain of submodules $\{ T^*(M) \}$.
of $M$ in the usual manner [1]. If $M \in 3$, $T^\alpha(M) = M$ for some ordinal $\alpha$ and the least such ordinal is the $T$-length of $M$.

For any module $M$, $M_i$ denotes the usual torsion subgroup of the group $M^+$ and $M_p$ for a prime $p$, denotes the maximum $p$-primary subgroup of $M^+$. Note that $M_i$ and $M_p$ are submodules of $M$.

We use $\text{Ext}(A, B)$ for $\text{Ext}^1_R(A, B)$ and $\text{Hom}(A, B)$ for $\text{Hom}_R(A, B)$. The reader is referred to MacLane [6] for the properties of Ext which are used in what follows.

2. The main theorem. The following lemma, in part characterizing the primary submodules of a module $M$, will be needed in the proof of the main theorem.

**Lemma 2.1.** Let $M \in 3$. Then

1. $M_i = M_{p_i}$ for $i \geq 1$.
2. $(M_0)^+$ is a torsion-free divisible group.

**Proof.** (1) Clearly $\text{Soc}(M_i) \subseteq M_{p_i}$ and by induction it is easy to see that $M_i \subseteq M_{p_i}$. But $M_i$ and $M_{p_i}$ are submodules of $M$ so $M_{p_i}/M_i$ is either zero or it has a simple submodule, since $M \in 3$. The latter choice leads to a contradiction, hence $M_i = M_{p_i}$.

(2) It is clear that $\text{Soc}(M_0)$ is divisible and torsion-free. Assume inductively that $T^\alpha(M_0)$ is divisible and torsion-free. Then since

$$T^{\alpha+1}(M_0)/T^\alpha(M_0) = \text{Soc}(M_0/T^\alpha(M_0))$$

is a direct sum of $Q$-type simples and since

$$0 \to T^\alpha(M_0) \to T^{\alpha+1}(M_0) \to T^{\alpha+1}(M_0)/T^\alpha(M_0) \to 0$$

is exact, we get that $T^{\alpha+1}(M_0)$ is torsion-free divisible, since the class of torsion-free divisible groups is closed under extensions. Since $M \in 3$, $T^\beta(M_0) = M_0$ for some $\beta$ and so $M_0$ is torsion-free divisible. This completes the proof.

**Remarks.** (1) From the previous lemma and the primary decomposition for torsion abelian groups, the primary decomposition holds for any ring which has no $Q$-type simple modules. In fact, over any ring, if $M \in 3$ and $M_0 = 0$, we get $M = \sum_{i=1}^\infty M_i$.

(2) For any module $M$, $\sum_{i \neq j} M_i$ cannot contain a simple from the class $s_1$ and consequently $\sum_{i=0}^\infty M_i$ is always a direct sum.

**Lemma 2.2.** The primary decomposition holds for the ring $R$ if and only if $\text{Ext}(S, T) = 0$ for each $Q$-type simple $S$ and each module $T \in 3$ with $T_0 = 0$.

**Proof.** The necessity of the condition is clear since if $0 \to T \to X \to S \to 0$ is exact and the primary decomposition holds, we
must have \( X = X_0 \oplus \sum_{i=1}^n X_i \) with \( X_0 \approx S \) and this implies that the sequence splits.

To prove that the condition is sufficient, let \( M \in \mathfrak{S} \). We will prove that \( M = \sum_{i=0}^n M_i \) by showing that \( M/\sum_{i=0}^n M_i \) has no simple submodule. By Lemma 2.1, \( \sum_{i=1}^n M_i = M_0 \), so \( \sum_{i=0}^n M_i = M_0 + M_t \). Also by 2.1, \( M_0 \) is divisible and so as groups we have

\[
\frac{M}{M_t} \approx \frac{M_0 + M_t}{M_t} \oplus \frac{K}{M_t}.
\]

Thus any subgroup of \( M/M_0 + M_t \) is isomorphic to a subgroup of \( K/M_t \). Hence \( M/M_0 + M_t \) contains no \( p \)-type simple since \( K/M_t \) is torsion-free as a group.

Assume \( M/M_0 + M_t \) has a \( Q \)-type simple, say \( S = X/M_0 + M_t \), \( X \subseteq M \). Now \( M_0 + M_t/M_0 \approx M_t \) so we have an exact sequence

\[
0 \to \frac{M_0 + M_t}{M_0} \approx M_t \to \frac{X}{M_0 + M_t} = S \to 0.
\]

By hypothesis, this sequence must split, so \( X/M_0 \) contains a \( Q \)-type simple. Thus \( M/M_0 \) contains a \( Q \)-type simple and this is a contradiction. Hence \( M/M_0 + M_t \) has no simple submodule, so \( M = \sum_{i=0}^n M_i \) and the proof is complete.

**Theorem 2.3.** The primary decomposition holds for the ring \( R \) if and only if

1. \( \prod_{S \in \mathfrak{S}} S/\sum_{S \in \mathfrak{S}} S \approx 0 \), where \( \mathfrak{C} \) is a representative set of simples of type \( p \).
2. If \( 0 \to P \to K \to U \to 0 \) is an exact sequence of \( R \)-modules with \( K \) cyclic, \( P \in \mathfrak{S}_i \) for some \( i \geq 1 \) and \( U \) a \( Q \)-type simple, then \( P \) has nonlimit ordinal \( T \)-length.

**Proof.** To see that the first condition is necessary, suppose \( U \) is a \( Q \)-type simple contained in the factor module \( \prod S/\sum S \). Then we have an exact sequence \( 0 \to \sum S \to X \to U \to 0 \), where \( X \subseteq \prod S \). Since the primary decomposition holds, \( X \approx U \oplus \sum S \). But then \( U \subseteq X \subseteq \prod S \) so

\[
0 \neq \text{Hom}(U, U) \subseteq \text{Hom}(U, \prod S) = \prod \text{Hom}(U, S) = 0
\]

and we have a contradiction.

If \( 0 \to P \to K \to U \to 0 \) is exact as in (2) above, then \( K \approx P \oplus U \) and so \( P \) is cyclic. It follows that \( P \) has nonlimit ordinal \( T \)-length.

To show that (1) and (2) are sufficient, we use Lemma 2.2 and show \( \text{Ext}(T, U) = 0 \) for \( U \) a \( Q \)-type simple and \( T \in \mathfrak{S}_i, T_0 = 0 \).
By previous remarks, \( T = \sum_{i=0}^{\alpha} T_i \) and applying \( \text{Hom}(U, -) \) to the exact sequence

\[
0 \rightarrow \sum T_i \rightarrow \prod T_i \rightarrow \prod T_i/\sum T_i \rightarrow 0
\]

we get the exact sequence

\[
\text{Hom}(U, \prod T_i/\sum T_i) \rightarrow \text{Ext}(U, \sum T_i) \rightarrow \text{Ext}(U, \prod T_i).
\]

But \( \text{Ext}(U, \prod T_i) = \prod \text{Ext}(U, T_i) \) so to show that the condition of Lemma 2.2 holds it is sufficient to prove:

(a) \( \text{Hom}(U, \prod T_i/\sum T_i) = 0 \),

(b) \( \text{Ext}(U, T_i) = 0 \) for \( i \geq 1 \).

By (1) and an easy modification of Lemma 2.2 of [2], (a) holds. We prove that (b) holds by showing \( \text{Ext}(U, A) = 0 \) for any \( p_i \)-primary module \( A \). The proof is by induction on the \( T \)-length of \( A \).

If \( A \) is a \( p_i \)-primary module of \( T \)-length one, then \( A = \sum S_{\alpha} \) where each \( S_{\alpha} \) is a \( p_i \)-type simple. As before

\[
\text{Hom}(U, \prod S_{\alpha}/\sum S_{\alpha}) \rightarrow \text{Ext}(U, \sum S_{\alpha}) \rightarrow \text{Ext}(U, \prod S_{\alpha})
\]

is exact with right end zero since \( \text{Ext}(U, S_{\alpha}) = 0 \) because \( U \) and \( S_{\alpha} \) are simples of different type. Since \( p_i U = U \), but \( p_i(\prod S_{\alpha}/\sum S_{\alpha}) = 0 \), we must have \( \text{Hom}(U, \prod S_{\alpha}/\sum S_{\alpha}) = 0 \). Hence \( \text{Ext}(U, A) = 0 \) if \( A \) has \( T \)-length one.

Now assume \( \text{Ext}(U, A) = 0 \) for all \( p_i \)-primary modules \( A \) of \( T \)-length \( \alpha < \beta \) and let \( B \) have \( T \)-length \( \beta \).

\[
(*) \quad 0 \rightarrow B \rightarrow X \rightarrow U \rightarrow 0
\]

is exact with \( B \rightarrow X \) the inclusion map, choose \( x \in X - B \). Then

\[
0 \rightarrow B \cap Rx \rightarrow Rx \rightarrow U \rightarrow 0
\]

is exact, the \( T \)-length of \( B \cap Rx \) is less than or equal to \( \beta \) and it is not a limit ordinal by (2). Let the \( T \)-length of \( B \cap Rx \) be \( \alpha + 1 \). Then

\[
0 \rightarrow B \cap Rx \rightarrow T^\alpha(B \cap Rx) \rightarrow Rx \rightarrow U \rightarrow 0
\]

is exact and since \( B \cap Rx / T^\alpha(B \cap Rx) \) has \( T \)-length one, the sequence must split. Thus there is a submodule \( K \) of \( Rx \) containing \( T^\alpha(B \cap Rx) \) with \( U \approx K / T^\alpha(B \cap Rx) \). Then

\[
0 \rightarrow T^\alpha(B \cap Rx) \rightarrow K \rightarrow U \rightarrow 0
\]

is exact and since \( T^\alpha(B \cap Rx) \) has \( T \)-length \( \alpha < \beta \), this sequence must split. Thus \( K \) contains a submodule isomorphic to \( U \) and so since
3. Applications and examples.

**Theorem 3.1.** Let $R$ be a ring with the property that every maximal left ideal $L$ of $R$, with $R/L$ a $Q$-type simple, is finitely generated. Then the primary decomposition holds for $R$.

**Proof.** We apply Theorem 2.3 and show that conditions (1) and (2) hold.

Let $0 \to P \to K \to U \to 0$ be exact with $K$ cyclic and $U$ a $Q$-type simple. Then for some left ideals $L \subseteq M$ of $R$ we have $P \approx M/L$ and $U \approx R/M$. But then $M$ is finitely generated so $M/L$, and hence $P$, cannot have limit ordinal $T$-length. Thus (2) holds.

Let $U = R(x_n + \sum S) \subseteq \prod S/ \sum S$ where the product and sum are taken over the set $C$ as in (1) of 2.3. Let $M = (\sum S_i(x_i))$. Then $U \approx R/M$, so $M = \text{Ann}_R P + \cdots + \text{Ann}_R U$ is finitely generated, since $U$ is of type $Q$. Now for each $i$, $m_i x_i = 0$ for all but finitely many $S \in C$. Hence there is an $x_{i_0} \neq 0$ such that $M x_{i_0} = 0$. But then $Rx_{i_0} \approx S_0 \approx R/M \approx U$ and this is a contradiction since $S_0$ is of type $p$. Hence (1) of 2.3 is satisfied and the proof is complete.

The following corollary generalizes part of Fuchs' Theorem 72.2 [5].

**Corollary 3.2.** Let $R$ satisfy the hypothesis of 3.1 and assume non-zero $R$-modules have nonzero socles. Then $R$ is the ring direct sum of two sided ideals $R_0, R_1, \ldots, R_n$ where $R_0^+$ is a direct sum of copies of $Q$ and each $R_i^+, 1 \leq i \leq n$, is a bounded primary group.

**Proof.** Since nonzero modules have nonzero socles, $R \in \Sigma$ and since by 3.1 the primary decomposition holds, we have $R = \sum_{i=0}^\infty R_i$. Since $R$ has a unit element, $R = R_0 \oplus \cdots \oplus R_n$. Each of the classes $R_i$ is closed under homomorphic images and right multiplication by elements of $R$ is a left $R$-homomorphism so each $R_i$ is a two-sided ideal. $R_0$ is a torsion-free divisible group and so it is a direct sum of copies of $Q$. Each $R_i, 1 \leq i \leq n$, is a primary group by Lemma 2.1 and since each $R_i$ is a ring with unit, it must be a bounded group. This proves the corollary.

To construct examples of non-Artinian rings satisfying the hypotheses of Corollary 3.2, let $P$ be an infinite product of copies of $Z_p$. Define the ring $R$ by $R^+ = P \oplus Z_p$ and $(p_1, i_1)(p_2, i_2) = (i_2 p_1 + i_1 p_2, i_1 i_2)$. Then $P$ is the socle of $R$ and $R/P \approx Z_p$; so nonzero modules have nonzero socles. Since $R$ has no $Q$-type simples, the hypothesis of 4.1 is clearly satisfied. Since every subgroup of $P$ is an ideal of $R$, it is clear that $R$ is neither Artinian or Noetherian.
References