ON POINTWISE PERIODIC TRANSFORMATION GROUPS

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Abstract. Let X be a connected and metrizable manifold without boundary, and \((X, T)\) a transformation group. We prove that if \(T\) is countable and pointwise periodic then \(T\) is periodic. This is a generalization of a result of Montgomery, which says that if \(h\) is a pointwise periodic homeomorphism of \(X\) onto itself then \(h\) is periodic.

We give in the following a generalization of a theorem of Montgomery (see [2]). For notation and terminology in the following see [1]. Throughout we denote by \(X\) a metric space with metric \(d\), and by \((X, T)\) a transformation group. We also assume that \(T\) is countable. We say that \(T = EA\) is a decomposition of \(T\) for \(x \in X\) if \(E, A\) are subsets of \(T\), \(xE = x\) and \(A\) is compact. \(T\) is periodic at \(x \in X\) if there is a decomposition of \(T\) for \(x\); \(T\) is pointwise periodic if it is periodic at each point of \(X\); \(T = EA\) is a decomposition of \(T\) for \(Y \subseteq X\) if it is a decomposition of \(T\) for each \(y\) in \(Y\); \(T\) is periodic if there is a decomposition of \(T\) for \(X\). The main result we wish to prove is the following.

Theorem A. Suppose \(X\) is a connected manifold without boundary. If \(T\) is pointwise periodic then it is periodic.

1. For the purposes of this article assume further that \(T\) is pointwise periodic. If \(T = EA\) is a decomposition of \(T\) for some \(x \in X\), then \(xT = xEA = xA\) is a compact and countable subset of \(X\). Hence some point of \(xA\) is an isolated point of \(xA\), and since \(T\) is transitive on \(xT\), each point of \(xA\) is an isolated point of \(xA\). Hence \(xA\), being compact, is finite. Let \(Z\) denote the set of all positive integers with the order topology, and define a function \(\phi: X \to Z\) by setting, for any \(x \in X\), \(x\phi\) to be the cardinality of \(xT\). We agree to denote a \(\delta\)-neighbourhood of \(x \in X\) by \(U(x, \delta)\).

Lemma 1. Let \(x \in X\) and \(T = EA\) be a decomposition of \(T\) for \(x\). Given any \(\epsilon > 0\), there exists a \(\delta > 0\), such that, for any \(f \in A\), \(U(x, \delta)f \subseteq U(x\epsilon, \epsilon)\) and for any \(f, g \in A\), if \(xf \neq xg\) then \(U(x, \delta)f \cap U(x, \delta)g = \emptyset\).

Proof. Let \(xT = xA = \{x_i: 1 \leq i \leq n\}\), and \(3\eta\) be the least of all the...
real numbers \( \{ d(x_i, x_j) : 1 \leq i, j \leq n, i \neq j \} \) and \( \epsilon \). Since \( A \) is a compact subset of \( T \), it is equicontinuous at \( x \). Hence there exists a \( \delta > 0 \), such that, for any \( f \in A \), \( U(x, \delta) \subseteq U(xf, \eta) \subseteq U(xf, \epsilon) \). If \( x_i = xf \neq xg = x_j \), for some \( f, g \) in \( A \), then \( d(x_i, x_j) \geq 3\eta \) and therefore \( U(x, \delta)f \cap U(x, \delta)g = \emptyset \). This completes the proof.

**Theorem 1.** \( \phi \) is lower semicontinuous on \( X \).

**Proof.** Let \( x \in X \), \( x\phi = n \in Z \), and \( xT = \{ x_i : 1 \leq i \leq n \} \). Let \( \epsilon = \min \{ d(x_i, x_j) : 1 \leq i, j \leq n, i \neq j \} \), and \( \delta > 0 \) be as in Lemma 1. Then, since for any \( y \in U(x, \delta) \), each \( U(x, \epsilon) \), \( 1 \leq i \leq n \), contains at least one element of \( yT \), \( y\phi \geq n \). This completes the proof.

**Theorem 2.** Suppose \( X \) is locally connected at \( x \in X \). If \( \phi \) is continuous at \( x \), then \( T \) is equicontinuous at \( x \).

**Proof.** Let \( \epsilon > 0 \) be given. Let \( T = EA \) be a decomposition of \( T \) for \( x \), \( xT = \{ x_i : 1 \leq i \leq n \} \), and \( A_i = \{ f \in A : xf = x_i \} \), \( 1 \leq i \leq n \). Since \( \phi \) is continuous at \( x \) there exists a \( \delta_1 > 0 \), such that, \( U(x, \delta_1)\phi = n \). Let \( \delta_2 > 0 \) be as in Lemma 1 with respect to \( \epsilon/2 \). Let \( V \) be a connected open set \( \subseteq U(x, \delta) \), and containing \( x \), where \( \delta = \min(\delta_1, \delta_2) \), and \( W_i = VA_i \), \( 1 \leq i \leq n \). Then \( \{ W_i : 1 \leq i \leq n \} \) is a disjoint family of open sets, and the diameter of each \( W_i \) is less than \( \epsilon \) (see Lemma 1).

Let \( y \in V \), and \( T = FB \) be a decomposition of \( T \) for \( y \). Then \( yA \subseteq yT = yB \). Since \( yA, i \subseteq W_i, 1 \leq i \leq n \), the cardinality of \( yA \geq n \). But since \( n \) is finite, and \( y\phi = n \), for \( \delta \leq \delta_1 \), we must have the cardinality of \( yA = n \). Hence \( yA = yB \). Thus \( yT = yB \subseteq U \{ W_i : 1 \leq i \leq n \} \). Since \( y \in V \) above was arbitrary, \( VT \subseteq U \{ W_i : 1 \leq i \leq n \} \).

Let \( f \in T \) be arbitrary. Then \( f = gh_i \), where \( g \in E \) and \( h_i \in A_i \), for some \( i \). Since \( Vf \) is connected, and \( Vf \cap W_i \neq \emptyset \) and \( \{ W_i : 1 \leq i \leq n \} \) is a disjoint family of open sets covering \( Vf \), it follows that \( Vf \subseteq W_i \). Since the diameter of \( W_i \) is less than \( \epsilon \), for any \( y \in V \), \( d(yf, xf) < \epsilon \).

Since \( V \) is open, it follows that \( T \) is equicontinuous at \( x \). This completes the proof.

**Lemma 2.** Let \( x \in X \) be a point of continuity of \( \phi \) and of equicontinuity of \( T \). Then there exists a \( \delta > 0 \), such that, for any two points of \( U(x, \delta) \) the decomposition of \( T \) for one is a decomposition of \( T \) for the other.

**Proof.** Let \( T = EA \) be a decomposition of \( T \) for \( x \), \( xT = \{ x_i : 1 \leq i \leq n \} \), and \( A_i = \{ f \in A : xf = x_i \} \), \( 1 \leq i \leq n \). Let \( 2\epsilon \) be the smallest of all \( d(x_i, x_j) \), \( 1 \leq i \leq n \), \( i \neq j \). Then from the continuity of \( \phi \) and the equicontinuity of \( T \) at \( x \), there exists a \( \delta > 0 \), such that, \( U(x, \delta)\phi = n \), and for any \( f \in T \), \( U(x, \delta)f \subseteq U(xf, \epsilon) \). Let \( U(x, \delta) = U \).
Let $f \in E$ and $y \in U$. If $yf \neq y$, then $U(x_i, \epsilon)$, where $x_i = x$, contains two distinct elements $y$ and $yf$. But each $U(x_i, \epsilon)$, $1 \leq i \leq n$, contains at least one element of $yT$, and any two of them are mutually disjoint, implying that $yT > n$. This is a contradiction since $y \in U$. Hence $yf = y$. Since $y \in U$ and $f \in E$ were arbitrary, $T = EA$ is a decomposition for $U$.

Let $W_i = UA_i$, $1 \leq i \leq n$. From the choice of $\epsilon$ and $\delta$ above, $W_i \cap W_j = \emptyset$ if $i \neq j$, $1 \leq i, j \leq n$. Let $f \in T$. Then from the decomposition $T = EA$, $f = hk$, where $h \in E$ and $k \in A$. But from the last paragraph above $h \in E$ implies that $h$ is the identity on $U$. Hence $f = k$ on $U$. Now if for some $y \in U$, $yf = y$, then $Uk \cap U \neq \emptyset$. Since $U \subseteq U(x, \epsilon)$ and $Uk \subseteq U(xk, \epsilon)$, it follows that $Uk \subseteq U(x, 2\epsilon)$.

Hence from the choice of $\epsilon$, $xk = x$, and consequently $xf = x$. This implies that any decomposition of $T$ for $y \in U$ is also a decomposition for $x$.

The above two results imply the statement of the lemma and complete the proof.

**Theorem 3.** Suppose $R$ is a connected open subset of $X$ on which $\phi$ is continuous and $T$ is equicontinuous. Then any decomposition of $T$ for any point of $R$ is a decomposition for $R$.

**Proof.** For each $x \in R$ there exists an $\eta(x)$ as in Lemma 2. Consider the open covering $\{ U(x, \eta(x)) : x \in R \}$ of $R$. Let $y$ be any given point of $R$, and $T = EA$ be a decomposition of $T$ for $y$. Since $R$ is connected, given any $z \in R$, there exists a finite set, $y = x_1, x_2, \ldots, x_n = z$ such that

$$U(x_i, \eta(x_i)) \cap U(x_{i+1}, \eta(x_{i+1})) \neq \emptyset, \quad 1 \leq i \leq n - 1.$$ 

Hence from Lemma 2 it follows, inductively, that $T = EA$ is a decomposition of $T$ for $z$. Since $z \in R$ was arbitrary the decomposition of $T$ for $y$ is a decomposition for $R$. This completes the proof.

The following is a consequence of Theorems 2 and 3.

**Theorem 4.** Suppose $X$ is locally connected, and $M$ is the set of all points of continuity of $\phi$. Then the decomposition of $T$ for any point of a component $R$ of $M$ is a decomposition for $R$. Indeed $T$ is periodic on each component of $M$.

2. **Proof of Theorem A.** Let $M = \{ x \in X : \phi$ is continuous at $x \}$. Then $M$ is everywhere dense in $X$, and since $Z$ is discrete, it is also open in $X$. Let $R$ be a component of $M$. Then from Theorem 4 there is a decomposition $T = EA$ of $T$ for $R$. Let $f \in E$. Since $T$ is periodic at each point $x \in X$, it is easy to see that $f$ is also periodic at each
point of $X$. Hence, from Montgomery's theorem \[2\], $f$ is periodic on $X$. But then Newman's Theorem 1 \[3\] implies, since $f$ is the identity on a connected open set $R$ in $X$, that $f$ is identity on $X$. Hence $T = EA$ is a decomposition of $T$ for $X$. This completes the proof.

**Remark.** Suppose $T$ is not necessarily countable as assumed above. Define $T$ to be *discretely periodic* at $x \in X$ if there exist subsets $E, A$ of $T$, such that, $T = EA$, $xE = x$ and $A$ is finite. We say that $T$ is discretely periodic if there exist subsets $E, A$ of $T$, such that, $T = EA$, $A$ is finite and $xE = x$ for each $x \in X$. Then each result in §1 above can be proved for $T$ which is discretely periodic at each $x \in X$, and consequently the following:

**Theorem B.** If $X$ is a connected manifold without boundary, and $T$ is discretely periodic at each point of $X$, then $T$ is discretely periodic.

**References**


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