NOTE ON THE EMBEDDING OF MANIFOLDS IN EUCLIDEAN SPACE

J. C. BECKER AND H. H. GLOVER

ABSTRACT. M. Hirsch and independently H. Glover have shown that a closed $k$-connected smooth $n$-manifold $M$ embeds in $\mathbb{R}^{2n-j}$ if $M_0$ immerses in $\mathbb{R}^{2n-j-1}$, $j \leq 2k$ and $2j \leq n-3$. Here $M_0$ denotes $M$ minus the interior of a smooth disk. In this note we prove the converse and show also that the isotopy classes of embeddings of $M$ in $\mathbb{R}^{2n-j}$ are in one-one correspondence with the regular homotopy classes of immersions of $M_0$ in $\mathbb{R}^{2n-j-1}$, $j < 2k - 1$ and $2j < n - 4$.

1. Introduction. Let $M$ be a closed $k$-connected smooth $n$-manifold and let $M_0$ denote $M$ minus the interior of a smooth disk. Hirsch [6] and Glover [2] have shown that $M$ embeds in $\mathbb{R}^{2n-j}$ if $M_0$ immerses in $\mathbb{R}^{2n-j-1}$, $j \leq 2k$ and $2j \leq n-3$. In this note we will apply the technique of [2] to prove

(1.1) Theorem. Suppose $j \leq 2k$ and $2j \leq n-3$. Then $M$ embeds in $\mathbb{R}^{2n-j}$ if and only if $M_0$ immerses in $\mathbb{R}^{2n-j-1}$.

(1.2) Theorem. Suppose $j \leq 2k - 1$ and $2j \leq n-4$. There is a one-one correspondence between the isotopy classes of embeddings of $M$ in $\mathbb{R}^{2n-1}$ and the regular homotopy classes of immersions of $M_0$ in $\mathbb{R}^{2n-j-1}$.

For $k > 0$, these extend results of Haefliger and Hirsch [5] over a range of $(k-1)$-dimensions.

Let $\nu$ denote a normal $m$-plane bundle over $M$, $m > n$, $\nu_0$ its restriction to $M_0$ and $\nu_0(j+1)$ the associated bundle with fibre $V_{m-m-n-j+1}$. According to Hirsch (see [7, Theorem 1.2]), the regular homotopy classes of immersions of $M_0$ in $\mathbb{R}^{2n-j-1}$ are in one-one correspondence with the vertical homotopy classes of sections to $\nu_0(j+1)$, which we denote by $C(\nu_0(j+1))$. In §5 we will show that $C(\nu_0(j+1)) \cong [M_0; V_{m-m-n-j+1}]$. Thus we have the following classification theorem.

(1.3) Corollary. Suppose $j \leq 2k - 1$ and $2j \leq n-4$. If $M$ embeds in $\mathbb{R}^{2n-j}$, the isotopy classes of embeddings of $M$ in $\mathbb{R}^{2n-j}$ are in one-one correspondence with the elements of $[M_0; V_{m-m-n-j+1}]$, $m > n$.

Received by the editors July 10, 1969.

AMS 1970 subject classifications. Primary 57D40.

Key words and phrases. Tubular neighborhood, deleted product, equivariant map, obstruction theory, Postnikov resolution.

1 The first author was supported by National Science Foundation Grant GP 24498, the second by Grant GP 5252.
2. The deleted product. Let $T(M)$ denote the tangent bundle of $M$ and $\Delta$ the diagonal of $M \times M$. The open disk of radius $r$ in $\mathbb{R}^n$ will be denoted by $D^r_\Delta$. Fix $p \in M$ and let $\phi: \mathbb{R}^n \to M$ be a diffeomorphism onto an open neighborhood of $p$ such that $\phi(0) = p$. Using a suitable partition of unity, construct a Riemannian metric $\rho$ on $T(M)$ such that $T(\phi): D^r_\Delta \times \mathbb{R}^n \to T(M)$ is metric preserving. Now choose $\epsilon$ so that $0 < \epsilon < 1/3$ and $E_\rho: T_\epsilon(M) \to M \times M$ by $E_\rho(v) = (\exp_\rho(v_2), \exp_\rho(-v_2))$ embeds $T_\epsilon(M)$ as a tubular neighborhood of the diagonal. Here $T_\epsilon(M)$ denotes the disk bundle of radius $\epsilon$. Let $U$, $V$ and $W$ denote the image under $\phi$ of $D^n_\epsilon$, $D^n_{2\epsilon}$ and $D^n_{3\epsilon}$ respectively. We may assume that $\epsilon$ is small enough that 

\[(2.1) \quad T_\epsilon(M) \cap (M \times U \cup \overline{U} \times M) \subseteq \phi(D^n_\epsilon) \times \phi(D^n_\epsilon).\]

Now the following properties are easily established.

\[(2.2) \quad T_\epsilon(M - V) \cap (M \times U \cup U \times M) = \Phi.\]

\[(2.3) \quad M \setminus V = M \times \{p\} \cup \{p\} \times M \text{ is a strong deformation retract of } M \times U \cup U \times M \cup \text{Int } T_\epsilon(V).\]

\[(2.4) \quad \Delta \cup \{p\} \times M \text{ is a strong deformation retract of } \text{Int } T_\epsilon(M) \cup U \times M \cup W \times U.\]

\[(2.5) \quad \Delta \cup (M \times \{p\}) \text{ is a strong deformation retract of } \text{Int } T_\epsilon(M) \cup M \times U \cup U \times W.\]

\[(2.6) \quad X = M \times M - (\text{Int } T_\epsilon(M) \cup U \times W \cup W \times U) \text{ is a strong deformation retract of } M \times M - \Delta.\]
Set \( A = X \cap (M \times U \cup U \times M) \) and note that (2.2) implies that \( S_\varepsilon(M - V) \subseteq X - A \), where \( S_\varepsilon(M - V) \) is the sphere bundle of radius \( \varepsilon \) over \( M - V \).

(2.7) Lemma. \( H^{2n-j}(X - A, S_\varepsilon(M - V)) = 0, j \leq 2k + 1 \).

Proof. We have

\[
H^{2n-j}(X - A, S_\varepsilon(M - V)) = H^{2n-j}(X - A \cup T_\varepsilon(M - V), T_\varepsilon(M - V))
\]

\[
= H^{2n-j}(X - A \cup T_\varepsilon(M - V)).
\]

Now note that

\[
X - A \cup T_\varepsilon(M - V) = M \times M - (M \times U \cup U \times M \cup \text{Int } T_\varepsilon(V)).
\]

Then by Poincaré duality and (2.3) we have

\[
H^{2n-j}(X - A \cup T_\varepsilon(M - V))
\]

\[
= H_j(M \times M, M \times U \cup U \times M \cup \text{Int } T_\varepsilon(V))
\]

\[
= H_j(M \times M, M \lor M) = 0, \quad j \leq 2k + 1.
\]

3. Equivariant maps. In this section we will record a few facts about equivariant maps which will be needed later. If \( X \) and \( Y \) are spaces with an involution, let \( E(X, Y) \) denote the set of equivariant homotopy classes of equivariant maps from \( X \) to \( Y \).

(3.1) Suppose \( Y \) is a finite CW-complex with a fixed point free cellular involution \( \alpha \). Let \( q \geq 0 \). There is a finite CW-complex \( Z \) with a fixed point free cellular involution and equivariant inclusion \( Y \subseteq Z \) such that

(a) \( Z \) is \((q-1)\)-connected,

(b) \( Z - Y \) consists of cells of dimension \( \leq q \).

Proof. Suppose \( Y \) is \((s-1)\)-connected. Let \([f_1] \cdots [f_t]\) generate \( \pi_s(Y) \) and let

\[
Z_1 = D^{s+1} \cup_{f_1} Y \cup_{\alpha f_1} D^{s+1}.
\]

Extend \( \alpha \) to an involution on \( Z_1 \) by interchanging the cells. Let \( i_1: Y \rightarrow Z_1 \) denote the inclusion. Then \( Z_1 \) is \((s-1)\)-connected and \( \pi_s(Z_1) \) is generated by \( i_{1*}[f_2] \cdots i_{1*}[f_t] \). Continue in this way.

(3.2) Let \( X \) and \( Y \) be finite CW-complexes with a fixed point free cellular involution and let \( f: X \rightarrow Y \) be equivariant. Suppose \( f^*: H^q(Y) \rightarrow H^q(X) \) is an isomorphism, \( q > t \), and is onto \( q = t \). Then \( f^*: E(Y, S^q) \rightarrow E(X, S^q) \) is one-one, \( q > t \), and is onto, \( q = t \).

This is well known from obstruction theory.

4. Proof of (1.1) and (1.2). Take \( Y \) to be \( S_\varepsilon(M - V) \) and \( q \) to be
\(2n - 2k - 2\) in (3.1) and let \(i: S_\epsilon(M - V) \to Z\) denote the inclusion. Because of statement (b) in (3.1) we have

\[(4.1) \text{Lemma. } i^*: H^{2n-j}(Z) \to H^{2n-j}(S_\epsilon(M - V)) \text{ is an isomorphism, } j \leq 2k + 1, \text{ and is onto, } j = 2k + 2.\]

By (2.7), \(i\) can be extended to an equivariant map

\[(4.2) \lambda: X - A \to Z.\]

We will now apply a construction of Glover [2]. Notice that

\[X = (X - A) \cup (U \times (M - W)) \cup ((M - W) \times U).\]

Let \(\Sigma(Z)\) denote the suspension of \(Z\) with the suspended involution and extend \(\lambda\) to an equivariant map

\[(4.3) \lambda: X \to \Sigma(Z)\]

by

\[\lambda(tv, y) = i^*(v, y) + (1 - t)s^+ \quad \text{and} \quad \lambda(y, tv) = i^*(y, v) + (1 - t)s^-\]

where \(v \in \text{Bdy}(U), y \in M - W, 0 \leq t \leq 1, \) and \(s^+, s^-\) are the north and south pole of \(\Sigma(Z)\) respectively.

\[(4.4) \text{Lemma. } \lambda^*: H^{2n+1-j}(\Sigma(Z)) \to H^{2n+1-j}(X) \text{ is an isomorphism, } j \leq 2k + 1, \text{ and is onto, } j = 2k + 2.\]

**Proof.** Let

\[X^+ = (X - A) \cup (U \times (M - W)), \quad X^- = (X - A) \cup ((M - W) \times U),\]

and let \(C^+(Z)\) and \(C^-(Z)\) denote the upper and lower cone in \(\Sigma(Z)\).

By (2.5)

\[(4.5) H^{2n-j}(X^+) \cong H_j(M \times M, \text{Int } T_\epsilon(M) \cup U \cup U \times W) \cong H_j(M \times M, \Delta \cup \{p\} \times M) = 0, \quad j \leq 2k + 1.\]

Similarly, by (2.6)

\[(4.6) H^{2n-j}(X^-) = 0, \quad j \leq 2k + 1.\]

By (2.7) and (4.1)

\[(4.7) \lambda^*: H^{2n-j}(Z) \to H^{2n-j}(X - A)\]
is an isomorphism, \( j \leq 2k + 1 \), and is onto, \( j = 2k + 2 \). The proof is now completed by comparing the Mayer-Vietoris sequence of \( \{ C^+(Z); C^-(Z) \} \) with that of \( \{ X^+; X^- \} \).

Now consider the maps

\[
\Sigma(S_+(M - V)) \xrightarrow{\Sigma(i)} \Sigma(Z) \xrightarrow{\lambda} X \subseteq M \times M - \Delta.
\]

Because of (2.6), \( X \) and \( M \times M - \Delta \) are equivariantly homotopy equivalent. Combining (4.1) and (4.4) with (3.2) and Theorem (2.5) of [1] we have

(4.9) Suppose \( j \leq 2k \). There is an equivariant map \( M \times M - \Delta \to S^{2n - j - 1} \) if and only if there is an equivariant map \( S_+(M - V) \to S^{2n - j - 2} \).

(4.10) Suppose \( j \leq 2k \). There is a one-one correspondence \( E(M \times M - \Delta, S^{2n - j}) \to E(S_+(M - V), S^{2n - j - 1}) \).

Theorems (1.1) and (1.2) now follow from the work of Haefliger [3] and Haefliger and Hirsch [4].

5. Proof of (1.3). This is proved by applying the following lemma to \( \nu_0(j + 1) \).

(5.1) Lemma. Suppose \( \beta = (E, B, p) \) is a fibration which admits a cross-section. Assume that \( B \) is \( k \)-connected and \( (n - k) \)-coconnected and the fibre \( F \) is \( (n - 2k - 1) \)-connected. Then \( C(\beta) \cong [B; F] \).

Proof. Set \( q = n - k - 1 \) and let \( \beta^{(q)} = (E^{(q)}, B, p^{(q)}) \) with fibre \( F^{(q)} \) denote the \( q \)th term in a Postnikov resolution of \( \beta \). Let \( \delta : B \to E^{(q)} \) be a cross-section and choose base-points \( b_0 \in B \) and \( x_0 = \delta(b_0) \in F^{(q)} \). The partial lifting \( g : B \vee F^{(q)} \to E^{(q)} \) of the projection \( \rho_1 : B \times F^{(q)} \to B \) defined by \( g(b, x_0) = \delta(b), b \in B \), and \( g(b_0, x) = x \), \( x \in F^{(q)} \), extends to a lifting \( \tilde{g} : B \times F^{(q)} \to E^{(q)} \) since \( H^r(B \times F^{(q)}, B \vee F^{(q)}; \pi_{r-1}(F^{(q)})) = 0, r \geq 0 \). Therefore \( \beta^{(q)} \) is weakly fibre homotopy equivalent to the product \( (B \times F^{(q)}, B, \rho_1) \) so that \( C(\beta^{(q)}) \cong [B; F^{(q)}] \).

Finally, since \( B \) is \( (q + 1) \)-coconnected \( C(\beta) \cong C(\beta^{(q)}) \) and \( [B; F] \cong [B; F^{(q)}] \).

References


University of Massachusetts, Amherst, Massachusetts 01003

Ohio State University, Columbus, Ohio 43210