LINEAR TRANSFORMATIONS UNDER WHICH THE DOUBLY STOCHASTIC MATRICES ARE INARIANT

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Abstract. Let \([\mathcal{M}_n(C)]\) denote the set of linear maps from the \(n \times n\) complex matrices into themselves and let \(\hat{\mathcal{O}}_n\) denote the set of complex doubly stochastic matrices, i.e. complex matrices whose row and column sums are 1. If \(F \in [\mathcal{M}_n(C)]\) is such that \(F(\hat{\mathcal{O}}_n) \subseteq \hat{\mathcal{O}}_n\) and \(F^*(\hat{\mathcal{O}}_n) \subseteq \hat{\mathcal{O}}_n\), then there exist \(A_i, B_i, A,\) and \(B \in \hat{\mathcal{O}}_n\) such that

\[
F(X) = \sum_i A_i X B_i + AX^tJ_n + J_nX'B - (1 + m)J_nXJ_n
\]

for all \(n \times n\) complex matrices \(X\), where \(J_n\) is the \(n \times n\) matrix whose elements are each \(1/n\) and where the superscript \(t\) denotes transpose. \(m\) denotes the number of the \(A_i\) (or \(B_i\)).

Introduction. It has been of considerable interest to study linear maps from the \(n \times n\) matrices to themselves that leave certain quantities invariant \([1] - [12]\). Often these maps are necessarily of the form \(F(X) = AXB\) or \(AX^tB\) with certain restrictions imposed on the \(n \times n\) matrices \(A\) and \(B\), where the superscript \(t\) denotes transpose. For example, Marcus and Moyls \([8]\) show that such maps which preserve spectral values are of these forms with \(A\) unimodular and \(B = A^{-1}\). They show in \([8], [9]\) that such maps which preserve certain given ranks are of these forms with \(A\) and \(B\) nonsingular. Marcus and May \([7]\) show that such maps which preserve the permanent function are of these forms with \(A = P_1D_1\) and \(B = P_2D_2\) where the \(P_i\) are permutation matrices and the \(D_i\) are diagonal matrices such that \(\text{per } D_1D_2 = 1\). Marcus, Minc, and Moyls \([10]\) show that one may assume that \(D_1 = D_2 = I\) if in addition the linear map leaves the doubly stochastic matrices invariant.

This paper is concerned with linear transformations which map the set of \(n \times n\) generalized doubly stochastic matrices, i.e. \(n \times n\) complex matrices whose row and column sums are one, into itself. It is shown that the set of such maps \(F\) which includes both \(F\) and \(F^*\) is precisely the set of linear combinations of transformations of the types \(AXB\) and \(CX^tD\), where the sum of the coefficients in any such
combination is one and \(A, B, C,\) and \(D\) are generalized doubly stochastic. It is clear that if \(F\) is such a combination, \(F(J_n) = F^*(J_n) = J_n,\) where \(J_n\) is the \(n \times n\) matrix whose entries are each \(1/n.\) There are linear maps not of this form which send the generalized doubly stochastic matrices into themselves which do not have \(J_n\) as a fixed point. For example, let \(F_1\) be the linear map from the \(2 \times 2\) complex matrices into themselves such that

\[
F_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a + b & 0 \\ 0 & c + d \end{array} \right).
\]

However, for such a map, the adjoint does not leave the generalized doubly stochastic matrices invariant.

We shall make use of the following notations and definitions. \(M_{mn}(C)\) shall denote the \(m \times n\) complex matrices, but we shall write \(M_n(C)\) in case \(m = n.\) \(0_m\) is the zero matrix in \(M_{mn}(C)\) whereas \(0_n\) and \(I_n\) are respectively the zero and identity matrix in \(M_n(C).\) \(E_{ij} \in M_{mn}(C)\) is a matrix whose element in the \((i, j)\)th position is 1 and whose elements are otherwise 0. \(M_n(C)\) will be given the usual inner product: \((X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij},\) where the bar denotes conjugation. The inner product induces the conventional norm on \(M_n(C): ||x||^2 = (X, X).\) \([M_n(C)]\) shall denote the set of linear maps of \(M_n(C)\) into itself. The lexicographic representation of \(X = (x_{ij}) \in M_n(C)\) is the column vector

\[
x = (x_{11} x_{12} \cdots x_{1n} x_{21} x_{22} \cdots x_{2n} \cdots x_{n1} x_{n2} \cdots x_{nn})^t.
\]

\(F_{nx}\) shall denote the \(n^2 \times n^2\) matrix representation of \(F \in [M_n(C)]\) such that \(F_{nx} x = y\) whenever \(F(X) = Y,\) where \(x\) and \(y\) are the lexicographic representations of \(X\) and \(Y,\) respectively. \(F_{nx}\) is called the faithful representation of \(F.\)

\(\hat{0}_n\) shall denote the \(n \times n\) generalized doubly stochastic matrices. If \(X_k \in M_{nk}(C), k = 1, \cdots, m,\) and \(n_1 + \cdots + n_m = n,\) the \(n \times n\) matrix

\[
X_1 \oplus X_2 \oplus \cdots \oplus X_m = \begin{bmatrix}
X_1 & 0 & \cdots & 0 \\
0 & X_2 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & X_m
\end{bmatrix}
\]

is called the direct sum of \(X_1, \cdots, X_m.\) The zeros indicate zero matrices of appropriate dimensions.

If \(X \in M_{mn}(C)\) and \(Y \in M_{pq}(C),\) the \(mp \times nq\) matrix
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\[ X \otimes Y = \begin{pmatrix}
    x_{11}Y & x_{12}Y & \cdots & x_{1n}Y \\
    x_{21}Y & x_{22}Y & \cdots & x_{2n}Y \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{m1}Y & x_{m2}Y & \cdots & x_{mn}Y \\
\end{pmatrix} \]

is called the Kronecker product of \( X \) and \( Y \).

The following well-known result is easily verified.

**Theorem 1.** Let \( A, B \in M_n(C) \), and let \( F \in [M_n(C)] \) be such that \( F(X) = AXB \) for all \( X \in M_n(C) \). Then \( F = A \otimes B^t \) is the faithful representation of \( F \).

**Preliminary results.**

**Lemma 1.** Let \( A \in M_{(pr,qs)}(C) \). There exist \( A_i \in M_{pq}(C) \) and \( B_i \in M_{rs}(C) \) such that \( A = \sum_i (A_i \otimes B_i) \).

**Proof.** Let the \( A_i \) be the matrices \( E_{ij} \in M_{pq}(C) \) listed in lexicographic order. Then write

\[ A = \begin{pmatrix}
    A_{11} & A_{12} & \cdots & A_{1q} \\
    A_{21} & A_{22} & \cdots & A_{2q} \\
    \vdots & \vdots & \ddots & \vdots \\
    A_{p1} & A_{p2} & \cdots & A_{pq} \\
\end{pmatrix}, \]

where each \( A_{ij} \in M_{rs}(C) \), and let the \( B_i \) be the \( A_{ij} \) arranged in lexicographic order. Clearly \( A = \sum_i (A_i \otimes B_i) \).

**Theorem 2.** Suppose that \( Q \in M_{pq}(C) \), \( R \in M_{rs}(C) \), and \( S \in M_{(pr,qs)}(C) \). There exist \( M_i \in M_{pq}(C) \) and \( N_i \in M_{rs}(C) \) such that \( Q = \sum_i M_i \), \( R = \sum_i N_i \), and \( S = \sum_i (M_i \otimes N_i) \).

**Proof.** By Lemma 1 there are \( A_i \in M_{pq}(C) \) and \( B_i \in M_{rs}(C) \) such that \( S - (Q \otimes R) = \sum_i (A_i \otimes B_i) \). Then

\[ Q = Q + \sum_i A_i + \sum_i 0_{pq} + \sum_i (-A_i), \]
\[ R = R + \sum_i B_i + \sum_i (-B_i) + \sum_i 0_{rs}, \]

and

\[ S = (Q \otimes R) + \sum_i (A_i \otimes B_i) + \sum_i (0_{pq} \otimes (-B_i)) + \sum_i ((-A_i) \otimes 0_{rs}). \]
Lemma 2. Let \( A, B \in \mathbb{M}_n(C) \), and let \( F \in [\mathbb{M}_n(C)] \) be such that \( F(X) = AXB \) for all \( X \in \mathbb{M}_n(C) \). Then \( F^*(X) = A^*XB^* \) for all \( X \in \mathbb{M}_n(C) \), where \( F^* \) is the adjoint of \( F \) and \( A^* \) and \( B^* \) are respectively the conjugate transposes of \( A \) and \( B \).

Proof. This follows from the fact that \( F^* = (A \otimes B^t)^* = A^* \otimes B^* \).

Lemma 3. Let \( U \) be a real unitary matrix in \( \mathbb{M}_n(C) \) with first column \( \left(1/\sqrt{n}\right)(1, 1, \ldots, 1)^t \). Define \( H \in [\mathbb{M}_n(C)] \) by \( H(X) = U^t X U \) for all \( X \in \mathbb{M}_n(C) \). Then \( H \) is unitary, and for each \( M \in \mathbb{H}_n \), there is an \( M' \in \mathbb{M}_{n-1}(C) \) such that \( H(M) = 1 \oplus M' \).

Proof. \( H \) is unitary by Lemma 2.

For \( M \in \mathbb{H}_n \) put \( W = H(M) \). Then

\[
W_{ij} = \sum_{i=1}^n \sum_{k=1}^n u_{ik} m_{ik} u_{kj} = \frac{1}{\sqrt{n}} \sum_{k=1}^n \sum_{i=1}^n m_{ik} u_{kj} = \frac{1}{\sqrt{n}} \sum_{k=1}^n u_{kj} = \delta_{ij},
\]

Kronecker's delta. Similarly, \( w_{11} = \delta_{11} \).

Lemma 4. Let \( P \in \mathbb{M}_{n^2}(C) \) be the permutation matrix such that for any \( A, B \in \mathbb{M}_{n-1}(C) \),

\[
P[(1 \oplus A) \otimes (1 \oplus B)]P^t = 1 \oplus A \oplus B \oplus (A \otimes B),
\]

and let \( T \) denote the transpose map. Then

\[
PT_{n^2}P^t = 1 \oplus \begin{pmatrix} 0_{n-1} & I_{n-1} \\ I_{n-1} & 0_{n-1} \end{pmatrix} \oplus T_{(n-1)^2}.
\]

Proof. Put

\[
V = 1 \oplus \begin{pmatrix} 0_{n-1} & I_{n-1} \\ I_{n-1} & 0_{n-1} \end{pmatrix} \oplus T_{(n-1)^2}.
\]

Let \( \sigma, \tau, \) and \( \omega \) respectively denote permutations of \( 1, \ldots, n^2 \) such that \( p_{i\sigma(i)} = t_{i\tau(i)} = v_{i\omega(i)} = 1 \) for \( i = 1, \ldots, n^2 \), where \( P = (p_{ij}), T_{n^2} = (t_{ij}), \) and \( V = (v_{ij}) \). Then

\[
\sigma(k) = (k - 1)n + 1, \quad k = 1, \ldots, n:
\]

\[
\sigma[(k(n - 1) + j)] = (k - 1)n + j, \quad j = 2, \ldots, n, \quad k = 1, \ldots, n,
\]

(1)

\[
\tau[(k(n - 1) + j)] = (j - 1)n + k, \quad j = 1, \ldots, n, \quad k = 1, \ldots, n,
\]

(2)

and
\[ \omega(1) = 1; \quad \omega(k) = n + k - 1, \quad k = 2, \cdots, n; \]
\[ \omega(n + j - 1) = j, \quad j = 2, \cdots, n; \]
\[ \omega[k(n - 1) + j + n] = j(n - 1) + k + n, \quad j = 1, \cdots, n - 1; \]
\[ k = 1, \cdots, n - 1. \]

If \( k = 1, \cdots, n \), \( \tau \sigma(k) = \tau [(k - 1)n + 1] = (1 - 1)n + k = k \); if \( j = 2, \cdots, n \), \( k = 1, \cdots, n \), \( \tau \sigma[k(n - 1) + j] = \tau [(k - 1)n + j] = (j - 1)n + k \). Also \( \sigma \omega(1) = \sigma(1) = 1 \), while if \( k = 2, \cdots, n \), \( \sigma \omega(k) = \sigma(n + k - 1) = (1 - 1)n + k = k \); if \( j = 2, \cdots, n \), \( \sigma \omega(n - 1 + j) = \sigma(j) = (j - 1)n + 1 \); if \( j = 2, \cdots, n \), \( k = 2, \cdots, n \),
\[ \sigma \omega[k(n - 1) + j] = \sigma \omega[(k - 1)(n - 1) + (j - 1) + n] \]
\[ = \sigma[(j - 1)(n - 1) + (k - 1) + n] \]
\[ = \sigma[j(n - 1) + k] = (j - 1)n + k. \]

Thus \( \tau \sigma(k) = \sigma \omega(k) \) for \( k = 1, \cdots, n^2 \), and therefore \( PT_{n^2} = VP \).

**Lemma 5.** If \( W \in \mathcal{O}_n \), then \( \| W - J_n \|^2 + 1 = \| W \|^2 \).

**Proof.** \( \| W - J_n \|^2 = (W - J_n, W - J_n) = (W, W) - (W, J_n) - (J_n, W) + (J_n, J_n) = \| W \|^2 - 1 - 1 + 1 = \| W \|^2 - 1. \)

It follows that for all \( W \in \mathcal{O}_n \), \( \| W \| \geq 1 \), and equality holds if and only if \( W = J_n \).

**Corollary.** If \( F \in [M_n(C)] \) is such that \( F(\mathcal{O}_n) \subseteq \mathcal{O}_n \) and \( F^*(\mathcal{O}_n) \subseteq \mathcal{O}_n \), then necessarily \( F(J_n) = F^*(J_n) = J_n \).

**Proof.** Suppose that \( F(J_n) = W \) and \( F^*(J_n) = X \). Put \( F(X) = Y \) and \( F^*(W) = Z \). Then \( W, X, Y, \) and \( Z \in \mathcal{O}_n \), and
\[ \| W \|^2 = (W, W) = (F(J_n), W) = (J_n, F^*(W)) = (J_n, Z) = 1; \]
whence \( W = J_n \). Likewise \( \| X \|^2 = (X, F^*(J_n)) = (Y, J_n) = 1 \), and so \( X = J_n \).

**Consequences.** Let \( K, L \in [M_n(C)] \) be defined respectively by \( K(X) = AXB \) and \( L(X) = AX^tB \), where \( A \) and \( B \in \mathcal{O}_n \) are fixed. Let \( U \) and \( H \) be as in Lemma 3. There exist matrices \( A', B' \in M_{n-1}(C) \) such that \( U'AU = 1 \oplus A' \) and \( U'BU = 1 \oplus B' \). Then, since \( H^*(X) = UXU^t \) for any \( X \in M_n(C) \),
\[ (HKH^*)(X) = (U'AU)X(U'BU) = (1 \oplus A')X(1 \oplus B'). \]

Thus \( (HKH^*)_{n^2} = (1 \oplus A') \otimes (1 \oplus B') \), and so
\[ P(HKH^*)_{n^2}P^t = 1 \oplus A' \oplus B' \oplus (A' \otimes B'), \]
where $P$ is as in Lemma 4.

Also if $T$ is the transpose map of Lemma 4,

$$(KLH^*)(X) = (HKT^*)(X) = (U^tAU)X'(U^tBU) = (HKH^T)(X).$$

Whence $(HLH^*)_{n^3} = (HKH^*)_{n^3}T_{n^3}$, and so, by Lemma 4,

$$P(HLH^*)_{n^3}P^t = P(HKH^*)_{n^3}P^tP^t_{n^3}$$

$$= (1 \oplus A \oplus B' \oplus (A' \oplus B'))
\cdot \left(1 \oplus \begin{pmatrix} 0_{n-1} & I_{n-1} \\ I_{n-1} & 0_{n-1} \end{pmatrix} \oplus T_{(n-1)^2} \right)
= 1 \oplus \begin{pmatrix} 0_{n-1} & A' \\ B' & 0_{n-1} \end{pmatrix} \oplus (A' \oplus B')T_{(n-1)^2}.$$

Note that the component $A' \otimes B'$ represents the reduced map $K'(Y) = A'YB'$ and $(A' \otimes B')T_{(n-1)^2}$ represents the reduced map $L'(Y) = A'Y'B'$, where $K', L' \in [M_{n-1}(C)]$.

Suppose that $F \in [M_{n}(C)]$ is such that $F(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$ and $F^*(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$.

By the corollary to Lemma 5, $F(J_n) = F^*(J_n) = J_n$. Since $H(J_n) = U'J_nU = 1 \oplus 0_{n-1}$, $(HFH^*)(1 \oplus 0_{n-1}) = 1 \oplus 0_{n-1}$.

Let $W, X \in \hat{\Omega}_n$ and put $Y = F(W), Z = F^*(X)$. There are matrices $W', X', Y', Z' \in M_{n-1}(C)$ such that $H(W) = 1 \oplus W', H(X) = 1 \oplus X', H(Y) = 1 \oplus Y'$, and $H(Z) = 1 \oplus Z'$. It follows that

$$(HFH^*)(1 \oplus W') = HF(W) = H(Y) = 1 \oplus Y'$$

and thus that

$$(HFH^*)(0 \oplus W') = (HFH^*) \{ (1 \oplus W') - (1 \oplus 0_{n-1}) \}
= (1 \oplus Y') - (1 \oplus 0_{n-1}) = 0 \oplus Y'.$$

Likewise, $(HFH^*)(1 \oplus X') = 1 \oplus Z'$ and $(HFH^*)(0 \oplus X') = 0 \oplus Z'$.

If $w'', x'', y''$ and $z''$ are the lexicographic representations of $1 \oplus W'$, $1 \oplus X'$, $1 \oplus Y'$, and $1 \oplus Z'$, and similarly for $w'''$, $x'''$, $y'''$, and $z'''$, then

$$(HFH^*)_{n^3}w'' = y'', \quad (HFH^*)_{n^3}x'' = z'',$

$$(HFH^*)_{n^3}w''' = y''', \quad (HFH^*)_{n^3}x''' = z''',$$

where $(HFH^*)_{n^3}$ is the conjugate transpose of $(HFH^*)_{n^3}$.

Note that $Pw'' = (1, \theta', w'')', Pw''' = (0, \theta', w''')'$, and similarly for $x'', x''', y'', y''', z'', z''', z''''$, where $\theta$ is a $2(n-1)$ dimensional column of zeros and where $w', x', y', z'$ are respectively the lexicographic representations of $W', X', Y', Z'$.

Put
\[
P(HFH^*)n^2P^t = \begin{bmatrix} F_0 & F_{12} & F_{13} \\ F_{21} & F_1 & F_{23} \\ F_{31} & F_{32} & F_2 \end{bmatrix},
\]

where \(F_0\) is \(1 \times 1\), \(F_1\) is \((n-1) \times 2(n-1)\), and \(F_2\) is \((n-1)^2 \times (n-1)^2\). Then

\[
\begin{bmatrix}
F_0 & F_{12} & F_{13} \\
F_{21} & F_1 & F_{23} \\
F_{31} & F_{32} & F_2
\end{bmatrix}
\begin{bmatrix}
1 \\
\theta \\
w'
\end{bmatrix}
=
\begin{bmatrix}
1 \\
\theta \\
w'
\end{bmatrix},
\]

\[
\begin{bmatrix}
F_0 & F_{12} & F_{13} \\
F_{21} & F_1 & F_{23} \\
F_{31} & F_{32} & F_2
\end{bmatrix}
\begin{bmatrix}
0 \\
\theta \\
w'
\end{bmatrix}
=
\begin{bmatrix}
0 \\
\theta \\
w'
\end{bmatrix},
\]

\[
\begin{bmatrix}
F_0 & F_{12} & F_{13} \\
F_{21} & F_1 & F_{23} \\
F_{31} & F_{32} & F_2
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
=
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}.
\]

The third equation in (1) indicates that \(F_{13}\) and \(F_{28}\) are zero; the fourth equation indicates that \(F_{31}\) and \(F_{28}\) are zero. Given these facts, the first equation indicates that \(F_0 = 1\) and \(F_{21} = \theta\). The second equation indicates that \(F_0 = 1\) and \(F_{13} = \theta^t\). Whence

\[
P(HFH^*)n^2P^t = 1 \oplus F_1 \oplus F_2.
\]

If we write

\[
F_1 = \begin{bmatrix} Q & Q' \\ R' & R \end{bmatrix},
\]

where \(Q, Q', R,\) and \(R' \in M_{n-1}(C)\), we have

\[
(2) \quad P(HFH^*)n^2P^t = (1 \oplus Q \oplus R \oplus F_2) + \left(0 \oplus \begin{bmatrix} 0_{n-1} & Q' \\ R'^t & 0_{n-1} \end{bmatrix} \oplus 0_{(n-1)^2}\right).
\]

Let \(L_1, L_2 \in \left[ M_n(C) \right] \) be respectively defined by \(L_1(X) = AX^tJ_n\) and \(L_2(X) = J_nX^tB\) where \(A = H^*(1 \oplus Q') \in \Omega_n\) and \(B = H^*(1 \oplus R') \in \Omega_n\). Then if \(J(X) = J_nXJ_n\) for all \(X \in M_n(C)\),

\[
\begin{bmatrix} 0_{n-1} & Q' \\ R'^t & 0_{n-1} \end{bmatrix} \oplus 0_{(n-1)^2} = \left(1 \oplus \begin{bmatrix} 0_{n-1} & Q' \\ R'^t & 0_{n-1} \end{bmatrix} \oplus (Q' \otimes 0_{(n-1)^t})
+ \left(1 \oplus \begin{bmatrix} 0_{n-1} & 0_{n-1} \\ R'^t & 0_{n-1} \end{bmatrix} \oplus (0_{n-1} \otimes R'^t) \right) T_{(n-1)^t}
\right)
- 2 \left(1 \oplus \begin{bmatrix} 0_{n-1} & 0_{n-1} \\ 0_{n-1} & 0_{n-1} \end{bmatrix} \oplus (0_{n-1} \otimes 0_{n-1}^t) \right).
\]

\[
(3) \quad = P(HL_1H^*)n^2P^t + P(HL_2H^*)n^2P^t - 2P(HJH^*)n^2P^t.
\]
By Theorem 2, there exist matrices $M_i$ and $N_i \in M_{n-1}(C)$ such that $Q = \sum_i M_i$, $R = \sum_i N_i$, and $F_2 = \sum_i (M_i \otimes N_i)$. For each $i$ let $A_i = H^*(1 \oplus M_i)$ and $B_i = H^*(1 \oplus N_i)$. Then each $A_i$ and $B_i \in \hat{\Omega}_n$. Let $K_i \in [M_n(C)]$ be defined by $K_i(X) = A_i X B_i$ for all $X \in M_n(C)$.

Then

$$1 \oplus Q \oplus R \oplus F_2 = \left( 1 \oplus \sum_i M_i \oplus \sum_i N_i \oplus \sum_i (M_i \otimes N_i) \right)$$

$$= \sum_i \left( 1 \oplus M_i \oplus N_i \oplus (M_i \otimes N_i) \right)$$

$$+ (1 - m)(1 \oplus 0_{n-1})$$

$$= \sum_i P(HK_iH^*)P + (1 - m)P(HJH^*)P$$

(4) where $m$ is the number of $M_i$ (or $N_i$).

It follows from (2), (3), and (4) that

$$F = \sum_i K_i + L_1 + L_2 - (1 + m)J.$$

In summary,

**Theorem 3.** Let $F \in [M_n(C)]$ be such that $F(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$ and $F^*(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$. Then there exist matrices $A_i, B_i, A$, and $B \in \hat{\Omega}_n$ such that

$$F(X) = \sum_i A_i X B_i + AX^tJ_n + J_n X^tB - (1 + m)J_n X J_n$$

for all $X \in M_n(C)$, where $m$ is the number of $A_i$ (or $B_i$).

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**References**


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