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LINEAR TRANSFORMATIONS UNDER WHICH THE DOUBLY STOCHASTIC MATRICES ARE INVARIANT[1]

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ABSTRACT. Let $[M_n(C)]$ denote the set of linear maps from the $n\times n$ complex matrices into themselves and let $\hat{\Omega}_n$ denote the set of complex doubly stochastic matrices, i.e. complex matrices whose row and column sums are 1. If $F\in[M_n(C)]$ is such that $F(\hat{\Omega}_n)\subseteq\hat{\Omega}_n$ and $F^*(\hat{\Omega}_n)\subseteq\hat{\Omega}_n$, then there exist A_i, B_i, A, and $B\in\hat{\Omega}_n$ such that

$$F(X)=\sum_i A_iXB_i+AX^tJ_n+J_nX^tB-(1+m)J_nXJ_n$$

for all $n\times n$ complex matrices X, where J_n is the $n\times n$ matrix whose elements are each $1/n$ and where the superscript t denotes transpose. m denotes the number of the A_i (or B_i).

Introduction. It has been of considerable interest to study linear maps from the $n\times n$ matrices to themselves that leave certain quantities invariant [1]–[12]. Often these maps are necessarily of the form $F(X)=AXB$ or AX^tB with certain restrictions imposed on the $n\times n$ matrices A and B, where the superscript t denotes transpose. For example, Marcus and Moyls [8] show that such maps which preserve spectral values are of these forms with A unimodular and $B=A^{-1}$. They show in [8], [9] that such maps which preserve certain given ranks are of these forms with A and B nonsingular. Marcus and May [7] show that such maps which preserve the permanent function are of these forms with $A=P_1D_1$ and $B=P_2D_2$ where the P_i are permutation matrices and the D_i are diagonal matrices such that per $D_1D_2=1$. Marcus, Minc, and Moyls [10] show that one may assume that $D_1=D_2=I$ if in addition the linear map leaves the doubly stochastic matrices invariant.

This paper is concerned with linear transformations which map the set of $n\times n$ generalized doubly stochastic matrices, i.e. $n\times n$ complex matrices whose row and column sums are one, into itself. It is shown that the set of such maps F which includes both F and F^* is precisely the set of linear combinations of transformations of the types AXB and CX^tD, where the sum of the coefficients in any such

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combination is one and A, B, C, and D are generalized doubly stochastic. It is clear that if F is such a combination, $F(J_n)=F^*(J_n)=J_n$, where J_n is the $n\times n$ matrix whose entries are each $1/n$. There are linear maps not of this form which send the generalized doubly stochastic matrices into themselves which do not have J_n as a fixed point. For example, let F_1 be the linear map from the 2×2 complex matrices into themselves such that

$$F_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}.$$

However, for such a map, the adjoint does not leave the generalized doubly stochastic matrices invariant.

We shall make use of the following notations and definitions. $M_{mn}(C)$ shall denote the $m\times n$ complex matrices, but we shall write $M_n(C)$ in case $m=n$. 0_{mn} is the zero matrix in $M_{mn}(C)$ whereas 0_n and I_n are respectively the zero and identity matrix in $M_n(C)$. $E_{ij}\in M_{mn}(C)$ is a matrix whose element in the (i, j)th position is 1 and whose elements are otherwise 0. $M_n(C)$ will be given the usual inner product: $(X, Y)=\sum_{i=1}^n\sum_{j=1}^n x_{ij}\bar{y}_{ij}$, where the bar denotes conjugation. The inner product induces the conventional norm on $M_n(C):\|x\|^2=(X,X)$. $[M_n(C)]$ shall denote the set of linear maps of $M_n(C)$ into itself. The lexicographic representation of $X=(x_{ij})\in M_n(C)$ is the column vector

$$x = (x_{11}\,x_{12}\cdots x_{1n}\,x_{21}\,x_{22}\cdots x_{2n}\cdots x_{n1}\,x_{n2}\cdots x_{nn})^t.$$

F_{n^2} shall denote the $n^2\times n^2$ matrix representation of $F\in[M_n(C)]$ such that $F_{n^2}x=y$ whenever $F(X)=Y$, where x and y are the lexicographic representations of X and Y, respectively. F_{n^2} is called the faithful representation of F.

$\hat{\Omega}_n$ shall denote the $n\times n$ generalized doubly stochastic matrices. If $X_k\in M_{n_k}(C)$, $k=1,\cdots,m$, and $n_1+\cdots+n_m=n$, the $n\times n$ matrix

$$X_1\oplus X_2\oplus\cdots\oplus X_m=\begin{pmatrix} X_1 & 0\cdots 0 & 0 \\ 0 & X_2\cdots 0 & 0 \\ \cdot & \cdot\quad\cdot\quad\cdot & \cdot \\ 0 & 0\cdots 0 & X_m \end{pmatrix}$$

is called the direct sum of $X_1,\cdots,X_m$. The zeros indicate zero matrices of appropriate dimensions.

If $X\in M_{mn}(C)$ and $Y\in M_{pq}(C)$, the $mp\times nq$ matrix

$$X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1n}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2n}Y \\ \cdot & \cdot & \cdot & \cdot \\ x_{m1}Y & x_{m2}Y & \cdots & x_{mn}Y \end{pmatrix}$$

is called the Kroneker product of X and Y.

The following well-known result is easily verified.

THEOREM 1. *Let A, $B \in M_n(C)$, and let $F \in [M_n(C)]$ be such that $F(X) = AXB$ for all $X \in M_n(C)$. Then $F_{n^2} = A \otimes B^t$ is the faithful representation of F.*

Preliminary results.

LEMMA 1. *Let $A \in M_{(pr)(qs)}(C)$. There exist $A_i \in M_{pq}(C)$ and $B_i \in M_{rs}(C)$ such that $A = \sum_i (A_i \otimes B_i)$.*

PROOF. Let the A_i be the matrices $E_{ij} \in M_{pq}(C)$ listed in lexicographic order. Then write

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \cdot & \cdot & \cdot & \cdot \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{pmatrix},$$

where each $A_{ij} \in M_{rs}(C)$, and let the B_i be the A_{ij} arranged in lexicographic order. Clearly $A = \sum_i (A_i \otimes B_i)$.

THEOREM 2. *Suppose that $Q \in M_{pq}(C)$, $R \in M_{rs}(C)$, and $S \in M_{(pr)(qs)}(C)$. There exist $M_i \in M_{pq}(C)$ and $N_i \in M_{rs}(C)$ such that $Q = \sum_i M_i$, $R = \sum_i N_i$, and $S = \sum_i (M_i \otimes N_i)$.*

PROOF. By Lemma 1 there are $A_i \in M_{pq}(C)$ and $B_i \in M_{rs}(C)$ such that $S - (Q \otimes R) = \sum_i (A_i \otimes B_i)$. Then

$$Q = Q + \sum_i A_i + \sum_i 0_{pq} + \sum_i (-A_i),$$

$$R = R + \sum_i B_i + \sum_i (-B_i) + \sum_i 0_{rs},$$

and

$$S = (Q \otimes R) + \sum_i (A_i \otimes B_i) + \sum_i (0_{pq} \otimes (-B_i)) + \sum_i ((-A_i) \otimes 0_{rs}).$$

LEMMA 2. *Let $A, B \in M_n(C)$, and let $F \in [M_n(C)]$ be such that $F(X) = AXB$ for all $X \in M_n(C)$. Then $F^*(X) = A^*XB^*$ for all $X \in M_n(C)$, where F^* is the adjoint of F and A^* and B^* are respectively the conjugate transposes of A and B.*

PROOF. This follows from the fact that $F_{n^2}^* = (A \otimes B^t)^* = A^* \otimes B^{*t}$.

LEMMA 3. *Let U be a real unitary matrix in $M_n(C)$ with first column $(1/\sqrt{n})\ (1, 1, \cdots, 1)^t$. Define $H \in [M_n(C)]$ by $H(X) = U^t X U$ for all $X \in M_n(C)$. Then H is unitary, and for each $M \in \hat{\Omega}_n$, there is an $M' \in M_{n-1}(C)$ such that $H(M) = 1 \oplus M'$.*

PROOF. H is unitary by Lemma 2.
For $M \in \hat{\Omega}_n$ put $W = H(M)$. Then

$$w_{1j} = \sum_{i=1}^{n} \sum_{k=1}^{n} u_{i1} m_{ik} u_{kj} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{i=1}^{n} m_{ik} u_{kj} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} u_{kj} = \delta_{1j},$$

Kroneker's delta. Similarly, $w_{i1} = \delta_{i1}$.

LEMMA 4. *Let $P \in M_{n^2}(C)$ be the permutation matrix such that for any $A, B \in M_{n-1}(C)$,*

$$P[(1 \oplus A) \otimes (1 \oplus B)]P^t = 1 \oplus A \oplus B \oplus (A \otimes B),$$

and let T denote the transpose map. Then

$$PT_{n^2}P^t = 1 \oplus \begin{pmatrix} 0_{n-1} & I_{n-1} \\ I_{n-1} & 0_{n-1} \end{pmatrix} \oplus T_{(n-1)^2}.$$

PROOF. Put

$$V = 1 \oplus \begin{pmatrix} 0_{n-1} & I_{n-1} \\ I_{n-1} & 0_{n-1} \end{pmatrix} \oplus T_{(n-1)^2}.$$

Let σ, τ, and ω respectively denote permutations of $1, \cdots, n^2$ such that $p_{i\sigma(i)} = t_{i\tau(i)} = v_{i\omega(i)} = 1$ for $i = 1, \cdots, n^2$, where $P = (p_{ij})$, $T_{n^2} = (t_{ij})$, and $V = (v_{ij})$. Then

$$\sigma(k) = (k-1)n + 1, \qquad k = 1, \cdots, n:$$

$$\sigma[k(n-1) + j] = (k-1)n + j, \tag{1}$$
$$j = 2, \cdots, n, \qquad k = 1, \cdots, n,$$

$$\tau[(k-1)n + j] = (j-1)n + k, \tag{2}$$
$$j = 1, \cdots, n, \qquad k = 1, \cdots, n,$$

and

$$
(3)\quad
\begin{aligned}
&\omega(1) = 1; \quad \omega(k) = n + k - 1, \quad k = 2, \cdots, n;\\
&\omega(n + j - 1) = j, \qquad\qquad j = 2, \cdots, n;\\
&\omega[k(n - 1) + j + n] = j(n - 1) + k + n,\\
&\qquad j = 1, \cdots, n - 1, \qquad k = 1, \cdots, n - 1.
\end{aligned}
$$

If $k=1, \cdots, n$, $\tau\sigma(k)=\tau[(k-1)n+1]=(1-1)n+k=k$; if $j=2, \cdots, n$, $k=1, \cdots, n$, $\tau\sigma[k(n-1)+j]=\tau[(k-1)n+j]=(j-1)n+k$. Also $\sigma\omega(1)=\sigma(1)=1$, while if $k=2, \cdots, n$, $\sigma\omega(k)=\sigma(n+k-1)=(1-1)n+k=k$; if $j=2, \cdots, n$, $\sigma\omega(n-1+j)=\sigma(j)=(j-1)n+1$; if $j=2, \cdots, n$, $k=2, \cdots, n$,

$$
\begin{aligned}
\sigma\omega[k(n - 1) + j] &= \sigma\omega[(k - 1)(n - 1) + (j - 1) + n]\\
&= \sigma[(j - 1)(n - 1) + (k - 1) + n]\\
&= \sigma[j(n - 1) + k] = (j - 1)n + k.
\end{aligned}
$$

Thus $\tau\sigma(k)=\sigma\omega(k)$ for $k=1, \cdots, n^2$, and therefore $PT_{n^2}=VP$.

LEMMA 5. *If $W\in\hat{\Omega}_n$, then $\|W-J_n\|^2+1=\|W\|^2$.*

PROOF. $\|W-J_n\|^2=(W-J_n, W-J_n)=(W, W)-(W, J_n)-(J_n, W_n)+(J_n, J_n)=\|W\|^2-1-1+1=\|W\|^2-1$.

It follows that for all $W\in\hat{\Omega}_n$, $\|W\|\geqq 1$, and equality holds if and only if $W=J_n$.

COROLLARY. *If $F\in[M_n(C)]$ is such that $F(\hat{\Omega}_n)\subseteq\hat{\Omega}_n$ and $F^*(\hat{\Omega}_n)\subseteq\hat{\Omega}_n$, then necessarily $F(J_n)=F^*(J_n)=J_n$.*

PROOF. Suppose that $F(J_n)=W$ and $F^*(J_n)=X$. Put $F(X)=Y$ and $F^*(W)=Z$. Then W, X, Y, and $Z\in\hat{\Omega}_n$, and

$$\|W\|^2 = (W, W) = (F(J_n), W) = (J_n, F^*(W)) = (J_n, Z) = 1;$$

whence $W=J_n$. Likewise $\|X\|^2=(X, F^*(J_n))=(Y, J_n)=1$, and so $X=J_n$.

Consequences. Let $K, L\in[M_n(C)]$ be defined respectively by $K(X)=AXB$ and $L(X)=AX^tB$, where A and $B\in\hat{\Omega}_n$ are fixed. Let U and H be as in Lemma 3. There exist matrices $A', B'\in M_{n-1}(C)$ such that $U^tAU=1\oplus A'$ and $U^tBU=1\oplus B'$. Then, since $H^*(X)=UXU^t$ for any $X\in M_n(C)$,

$$(HKH^*)(X) = (U^tAU)X(U^tBU) = (1 \oplus A')X(1 \oplus B').$$

Thus $(HKH^*)_{n^2}=(1\oplus A')\otimes(1\oplus B'^t)$, and so

$$P(HKH^*)_{n^2}P^t = 1 \oplus A' \oplus B'^t \oplus (A' \otimes B'^t),$$

where P is as in Lemma 4.

Also if T is the transpose map of Lemma 4,

$$(KLH^*)\,(X) = (HKTH^*)\,(X) = (U^tAU)\,X^t(U^tBU) = (HKH^*T)\,(X).$$

Whence $(HLH^*)_{n^2} = (HKH^*)_{n^2}T_{n^2}$, and so, by Lemma 4,

$$\begin{aligned} P(HLH^*)_{n^2}P^t &= P(HKH^*)_{n^2}P^tPT_{n^2}P^t \\ &= (1 \oplus A' \oplus B'^t \oplus (A' \otimes B'^t)) \\ &\quad \cdot \left(1 \oplus \begin{pmatrix} 0_{n-1} & I_{n-1} \\ I_{n-1} & 0_{n-1} \end{pmatrix} \oplus T_{(n-1)^2}\right) \\ &= 1 \oplus \begin{pmatrix} 0_{n-1} & A' \\ B'^t & 0_{n-1} \end{pmatrix} \oplus (A' \otimes B'^t)T_{(n-1)^2}. \end{aligned}$$

Note that the component $A'\otimes B'^t$ represents the reduced map $K'(Y) = A'YB'$ and $(A'\otimes B'^t)T_{(n-1)^2}$ represents the reduced map $L'(Y) = A'Y^tB'$, where $K', L' \in [M_{n-1}(C)]$.

Suppose that $F \in [M_n(C)]$ is such that $F(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$ and $F^*(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$. By the corollary to Lemma 5, $F(J_n) = F^*(J_n) = J_n$. Since $H(J_n) = U^tJ_nU = 1 \oplus 0_{n-1}$, $(HFH^*)\,(1 \oplus 0_{n-1}) = 1 \oplus 0_{n-1}$.

Let W, $X \in \hat{\Omega}_n$ and put $Y = F(W)$, $Z = F^*(X)$. There are matrices W', X', Y', and $Z' \in M_{n-1}(C)$ such that $H(W) = 1 \oplus W'$, $H(X) = 1 \oplus X'$, $H(Y) = 1 \oplus Y'$, and $H(Z) = 1 \oplus Z'$. It follows that

$$(HFH^*)(1 \oplus W') = HF(W) = H(Y) = 1 \oplus Y'$$

and thus that

$$\begin{aligned} (HFH^*)(0 \oplus W') &= (HFH^*)\{(1 \oplus W') - (1 \oplus 0_{n-1})\} \\ &= (1 \oplus Y') - (1 \oplus 0_{n-1}) = 0 \oplus Y'. \end{aligned}$$

Likewise, $(HFH^*)^*\,(1 \oplus X') = 1 \oplus Z'$ and $(HFH^*)^*\,(0 \oplus X') = 0 \oplus Z'$.

If w'', x'', y'' and z'' are the lexicographic representations of $1 \oplus W'$, $1 \oplus X'$, $1 \oplus Y'$, and $1 \oplus Z'$, and similarly for w''', x''', y''', z''' and $0 \oplus W'$, $0 \oplus X'$, $0 \oplus Y'$, $0 \oplus Z'$, then

$$\begin{aligned} (HFH^*)_{n^2}w'' &= y'', \qquad & (HFH^*)^*_{n^2}x'' &= z'', \\ (HFH^*)_{n^2}w''' &= y''', \qquad & (HFH^*)^*_{n^2}x''' &= z''', \end{aligned}$$

where $(HFH^*)^*_{n^2}$ is the conjugate transpose of $(HFH^*)_{n^2}$.

Note that $Pw'' = (1, \theta^t, w'^t)^t$, $Pw''' = (0, \theta^t, w'^t)^t$, and similarly for x'', x''', y'', y''', and z'', z''', where θ is a $2(n-1)$ dimensional column of zeros and where w', x', y', and z' are respectively the lexicographic representations of W', X', Y', and Z'.

Put

$$P(HFH^*)_{n^2}P^t = \begin{pmatrix} F_0 & F_{12} & F_{13} \\ F_{21} & F_1 & F_{23} \\ F_{31} & F_{32} & F_2 \end{pmatrix},$$

where F_0 is 1×1, F_1 is $2(n-1)\times 2(n-1)$, and F_2 is $(n-1)^2\times(n-1)^2$. Then

$$(1)\qquad \begin{aligned} &\begin{pmatrix} F_0 & F_{12} & F_{13} \\ F_{21} & F_1 & F_{23} \\ F_{31} & F_{32} & F_2 \end{pmatrix}\begin{pmatrix} 1 \\ \theta \\ w' \end{pmatrix} = \begin{pmatrix} 1 \\ \theta \\ y' \end{pmatrix}, \qquad \begin{pmatrix} F_0 & F_{12} & F_{13} \\ F_{21} & F_1 & F_{23} \\ F_{31} & F_{32} & F_2 \end{pmatrix}^*\begin{pmatrix} 1 \\ \theta \\ x' \end{pmatrix} = \begin{pmatrix} 1 \\ \theta \\ z' \end{pmatrix}, \\ &\begin{pmatrix} F_0 & F_{12} & F_{13} \\ F_{21} & F_1 & F_{23} \\ F_{31} & F_{32} & F_2 \end{pmatrix}\begin{pmatrix} 0 \\ \theta \\ w' \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \\ y' \end{pmatrix}, \qquad \begin{pmatrix} F_0 & F_{12} & F_{13} \\ F_{21} & F_1 & F_{23} \\ F_{31} & F_{32} & F_2 \end{pmatrix}^*\begin{pmatrix} 0 \\ \theta \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \\ z' \end{pmatrix}. \end{aligned}$$

The third equation in (1) indicates that F_{13} and F_{23} are zero; the fourth equation indicates that F_{31} and F_{32} are zero. Given these facts, the first equation indicates that $F_0=1$ and $F_{21}=\theta$. The second equation indicates that $F_0=1$ and $F_{12}=\theta^t$. Whence

$$P(HFH^*)_{n^2}P^t = 1 \oplus F_1 \oplus F_2.$$

If we write

$$F_1 = \begin{pmatrix} Q & Q' \\ R'^t & R \end{pmatrix}$$

where Q, Q', R, and $R' \in M_{n-1}(C)$, we have

$$(2)\quad P(HFH^*)_{n^2}P^t = (1\oplus Q\oplus R\oplus F_2) + \left(0\oplus\begin{pmatrix} 0_{n-1} & Q' \\ R'^t & 0_{n-1} \end{pmatrix}\oplus 0_{(n-1)^2}\right).$$

Let $L_1, L_2 \in [M_n(C)]$ be respectively defined by $L_1(X) = AX^tJ_n$ and $L_2(X) = J_nX^tB$ where $A = H^*(1\oplus Q') \in \hat{\Omega}_n$ and $B = H^*(1\oplus R') \in \hat{\Omega}_n$. Then if $J(X) = J_nXJ_n$ for all $X \in M_n(C)$,

$$(3)\quad \begin{aligned} \left(0\oplus\begin{pmatrix} 0_{n-1} & Q' \\ R'^t & 0_{n-1} \end{pmatrix}\oplus 0_{(n-1)^2}\right) &= \left(1\oplus\begin{pmatrix} 0_{n-1} & Q' \\ 0^t_{n-1} & 0_{n-1} \end{pmatrix}\oplus(Q'\otimes 0^t_{n-1})T_{(n-1)^2}\right) \\ &\quad + \left(1\oplus\begin{pmatrix} 0_{n-1} & 0_{n-1} \\ R'^t & 0_{n-1} \end{pmatrix}\oplus(0_{n-1}\otimes R'^t)T_{(n-1)^2}\right) \\ &\quad - 2\left(1\oplus\begin{pmatrix} 0_{n-1} & 0_{n-1} \\ 0_{n-1} & 0^t_{n-1} \end{pmatrix}\oplus(0_{n-1}\otimes 0^t_{n-1})\right) \\ &= P(HL_1H^*)_{n^2}P^t + P(HL_2H^*)_{n^2}P^t - 2P(HJH^*)_{n^2}P^t. \end{aligned}$$

By Theorem 2, there exist matrices M_i and $N_i \in M_{n-1}(C)$ such that $Q=\sum_i M_i$, $R=\sum_i N_i^t$, and $F_2=\sum_i (M_i \otimes N_i^t)$. For each i let $A_i = H^*(1 \oplus M_i)$ and $B_i = H^*(1 \oplus N_i)$. Then each A_i and $B_i \in \hat{\Omega}_n$. Let $K_i \in [M_n(C)]$ be defined by $K_i(X) = A_i X B_i$ for all $X \in M_n(C)$.

Then

$$
\begin{aligned}
1 \oplus Q \oplus R \oplus F_2 &= \left(1 \oplus \sum_i M_i \oplus \sum_i N_i^t \oplus \sum_i (M_i \otimes N_i^t)\right) \\
&= \sum_i (1 \oplus M_i \oplus N_i^t \oplus (M_i \otimes N_i^t)) \\
&\quad + (1-m)(1 \oplus 0_{n^2-1}) \\
&= \sum_i P(HK_iH^*)_{n^2}P^t + (1-m)P(HJH^*)_{n^2}P^t
\end{aligned} \tag{4}
$$

where m is the number of M_i (or N_i).

It follows from (2), (3), and (4) that

$$F = \sum_i K_i + L_1 + L_2 - (1+m)J.$$

In summary,

THEOREM 3. *Let $F \in [M_n(C)]$ be such that $F(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$ and $F^*(\hat{\Omega}_n) \subseteq \hat{\Omega}_n$. Then there exist matrices A_i, B_i, A, and $B \in \hat{\Omega}_n$ such that*

$$F(X) = \sum_i A_i X B_i + AX^t J_n + J_n X^t B - (1+m) J_n X J_n$$

for all $X \in M_n(C)$, where m is the number of A_i (or B_i).

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