A NOTE ON FOURIER MULTIPLIERS

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Abstract. A short proof is given of de Leeuw's restriction result for multipliers.

In this note we prove directly the following result of de Leeuw (Proposition 3.2 in [1]).

Theorem. Let $m(x, y)$ be a Fourier multiplier for $L^p(R^{i+j})$. Then for almost every $x$, $m_x(y) = m(x, y)$ is a Fourier multiplier for $L^p(R^j)$ and the multiplier norm of $m_x$ does not exceed that of $m$. In particular, the restriction is possible for each $x$ such that $(x, y)$ is a Lebesgue point of $m$ for almost all $y \in R^j$.

To prove this we recall that the (necessarily) bounded measurable function $m$ is a Fourier multiplier for $L^p(R^n)$ if and only if there is a constant $C$ such that, for $f, g \in C_0^\infty(R^n)$,

\[ |m(x)| \leq (2\pi)^n C \|f\|_p \|g\|_{p'}, \]

where $f(x) = \int f(y) e^{-ix \cdot y} dy$ and $1/p + 1/p' = 1$. The best constant $C$ is then the norm of the operator $K$ defined by $m \hat{=} (Kf)^\wedge$, and we write $\|m\|_p$ for this quantity.

Remark. The inequality (*) might be taken as the definition of Fourier multiplier instead of: $m(L^p) \subseteq (L^p)\wedge$ for $1 \leq p \leq 2$, a duality argument for $p > 2$.

If $p = 1$ or $p = 2$ the result is obvious since Fourier transforms of $L^1$ functions restrict and since the Fourier multipliers for $L^2$ are the essentially bounded measurable functions.

In the other cases let $f, \varphi \in C_0^\infty(R^n)$, $g, \psi \in C_0^\infty(R^j)$. We assume at first that $m$ is continuous. Apply (*) and Fubini, using $f(x, y)g(y)$ for $f(x, y)$, $\varphi(x)\psi(y)$ for $g(x, y)$, to deduce that

\[ I(x) = \frac{1}{(2\pi)^i} \int m(x, y)\hat{g}(y)\hat{\psi}(y)dy \]

is a Fourier multiplier for $L^p(R^i)$, with $\|I\|_p \leq \|m\|_p \|g\|_p \|\psi\|_{p'}$. Since

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\[ |m\|_\infty \leq |m|_p \text{ (for } |m|_p = |m|_{p'}, \quad |m|_\infty = |m|_2 \leq |m|_{p'}^{1/2} |m|_{p''}^{1/2} \text{ by the Riesz interpolation theorem), we have} \]

\[ |I(x)| \leq |m|_{p'} \|g\|_{p'} \|\psi\|_{p'}, \]

which is (*) for \( m_z \).

To remove the restriction of continuity, form the convolution \( a_\epsilon m \) of \( m \) with \( \epsilon^{-n} \) times the characteristic function of the \( \epsilon \)-cube centered at the origin. A change in the order of integration in (*) gives \( |a_\epsilon m|_p \leq |m|_p \). When \((x, y)\) is a Lebesgue point of \( m \) for almost all \( y \) in \( \mathbb{R}^l \), \( (a_\epsilon m)_z \rightarrow m_z \) pointwise and boundedly as \( \epsilon \rightarrow 0 \). From (*), the preceding paragraph, and dominated convergence it follows that \( m_z \) is a Fourier multiplier for \( L^p(\mathbb{R}^l) \), and \( |m_z|_p \leq |m|_p \).

**Reference**


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