TWO CELLS WITH \( N \) POINTS OF LOCAL NONCONVEXITY

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Abstract. A subset \( S \) of the plane is a two cell provided \( S \) is homeomorphic to \( \{ x \mid \| x \| \leq 1 \} \).

Theorem. Let \( S \) be a two cell with exactly \( n \) points of local nonconvexity. Then \( S \) is expressible as a union of \( n + 1 \) compact convex sets with mutually disjoint interiors.

1. Introduction. We will prove that a two cell \( S \) in \( \mathbb{R}^2 \) which has exactly \( n \) points of local nonconvexity is expressible as a union of \( n + 1 \) compact convex sets with mutually disjoint interiors. It follows immediately that \( S \) is \( n + 1 \) polygonally connected, i.e., an \( L_{n+1} \) set and thus we have as a corollary a special case of a result of Valentine [1].

Throughout, the symbols \( \cup, \cap \) and \( \sim \) denote set union, set intersection and set difference respectively. The interior, closure, and boundary of a set \( S \subset \mathbb{R}^2 \) are denoted by \( \text{int} \ S \), \( \text{cl} \ S \) and \( \text{bd} \ S \), respectively. The convex hull of a set \( S \subset \mathbb{R}^2 \) is denoted by \( H(S) \). If \( x, y \in S \) then \( [xy] \) and \( (xy) \) denote the closed and open line segments joining \( x \) to \( y \), respectively. The Euclidean norm is given by \( \| \| \) and \( d \) denotes the Hausdorff metric on compact subsets of the plane as given in Valentine [2]. If \( S \) is a set, \( |S| \) denotes its cardinality. We define the distance between a point \( x \) and a set \( S \) as \( \inf \{ \| x - y \| \mid y \in S \} \); we denote this distance by \( p(x, S) \). By the \( \epsilon \) ball about a set \( S \) we mean \( \{ x \mid p(x, S) < \epsilon \} \).

Definition 1. A point \( x \in S \) is called a point of local convexity of \( S \) if there exists a neighborhood \( N \) of \( x \) such that \( N \cap S \) is convex. If such a neighborhood does not exist, \( x \) is called a point of local nonconvexity.

Definition 2. A set \( S \) is said to be starshaped relative to a point \( y \) if for each \( x \in S \), \( [xp] \subset S \).

Definition 3. A segment \( [xy] \) is said to be a crosscut of a set \( S \) provided \( x, y \in \text{bd} \ S \) and \( (xy) \subset \text{int} \ S \).

Definition 4. A set \( S \subset \mathbb{R}^2 \) is a two cell provided \( S \) is homeomorphic to \( \{ x \mid \| x \| \leq 1 \} \).

Given a two cell \( S \) and a crosscut \( [xy] \) of \( S \), \( [xy] \) induces a natural decomposition of \( S \) into two new two cells \( D^1xy \) and \( D^2xy \) such that
For a proof of this, see Newman [1]. Specifically, the points \( x \) and \( y \) divide \( \partial S \) into two disjoint relatively open connected subsets of \( \partial S \), say \( C^1xy \) and \( C^2xy \), and \( D^1xy \) is the set whose boundary is given by \( [xy] \cup C^1xy \) and \( D^2xy \) is the set whose boundary is given by \( [xy] \cup C^2xy \). If \( S \) is a set, \( C(S) \) and \( L(S) \) will denote its points of local convexity and nonconvexity, respectively, and \( \emptyset \) will denote the empty set. The following two theorems constitute the main results of this paper.

Theorem 1. Let \( S \subseteq \mathbb{R}^2 \) be a two cell such that \( |L(S)| = n, n \geq 2 \). Then there exists a crosscut \( [xy] \) of \( S \) such that \( x, y \in C(S) \) and \( |L(D^i xy)| \leq n - 1 \) for \( i = 1, 2 \).

Theorem 2. Let \( S \subseteq \mathbb{R}^2 \) be a two cell such that \( |L(S)| = n \). Then \( S = \bigcup_{i=1}^{n+1} C_i \), where \( C_i, 1 \leq i \leq n + 1 \), are compact, convex and have mutually disjoint interiors.

Theorem 1 is of some independent interest, since it makes induction arguments readily accessible. Also, note that if \( n = 0 \) in Theorem 2, then the latter reduces to a special case of the important theorem of Tietze [1], which we state for later reference.

Theorem 3. Let \( S \) be a closed connected set in \( \mathbb{R}^d \), all of whose points are points of local convexity. Then \( S \) is convex.

We also shall utilize the following result of Valentine [1].

Theorem 4. Let \( S \) be a closed connected set in \( \mathbb{R}^d \) such that \( L(S) \) is not empty. Then given \( x \in S \), there exists \( y \in L(S) \) such that \( [xy] \subseteq S \).

II. Preliminary results and proof of Theorems 1 and 2.

Theorem 5. Let \( \{A_i\}_{i=1}^{\infty} \) be a sequence of compact sets in \( \mathbb{R}^2 \) converging to a set \( A \) in the Hausdorff metric \( \delta \) such that for each \( i \), \( A_i = \bigcup_{j=1}^{n+1} C_j^i \) where each \( C_j^i \) is convex and compact for \( 1 \leq j \leq n + 1 \) and \( \text{int } C_j^i \cap \text{int } C_k^i = \emptyset \), for \( k \neq j \) and \( 1 \leq k, j \leq n + 1 \). Then \( A = \bigcup_{j=1}^{n+1} C_j \) where \( C_j \) is compact and convex for \( 1 \leq j \leq n + 1 \) and \( \text{int } C_j \cap \text{int } C_k = \emptyset \) for \( j \neq k, 1 \leq j, k \leq n + 1 \).

Proof. By the theorem of Blaschke we may assume without loss of generality that for each \( j \) the sequence \( \{C_j^i\}_{i=1}^{\infty} \) converges to a set \( C_j \). Now \( C_j \) is compact and convex, since \( C_j \) is the limit of compact convex sets. Since for each \( i \) and \( j \neq k \) \( C_j \cap \text{int } C_k = \emptyset \), we have \( \text{int } C_j \cap \text{int } C_k = \emptyset \), and this completes the proof.

Theorem 6. Let \( S \subseteq \mathbb{R}^2 \) be a two cell such that \( |L(S)| = 1 \). Then \( S = C_1 \cup C_2 \) where \( C_1 \) and \( C_2 \) are compact, convex and \( \text{int } C_1 \cap \text{int } C_2 = \emptyset \).
Proof. Throughout the proof let \( L(S) = \{x\} \).

Case 1. Suppose there exist \( z, q \in \text{bd } S \), such that \( z \neq x, q \neq x \), \( [zx] \cup [xq] \subseteq \text{bd } S \) and \( [zx] \cap [xq] = x \). Then clearly we may choose a crosscut \([xy]\) where \( y \in C(S) \) such that \( x \in C(D^1xy) \cap C(D^2xy) \). Since \( y \in C(S) \), we have \( y \in C(D^1xy) \cap C(D^2xy) \). Then \( D^1xy \) and \( D^2xy \) are convex by Tietze's Theorem and \( \text{int } D^1xy \cap \text{int } D^2xy = \emptyset \) by definition. Thus \( D^1xy \) and \( D^2xy \) are the required sets.

Case 2. Suppose for each \( y \in \text{bd } S, y \neq x \), that \([xy] \subseteq \text{bd } S \). For this case, we need a lemma.

Lemma 1. Let \( S \) be a two cell satisfying Case 2. Then if \( t \in \text{bd } S, t \neq x \), \( (xt) \subseteq \text{int } S \).

Proof. Suppose the lemma is false. By Theorem 4, \( S \) is starshaped relative to \( x \), so \([xt] \subseteq \text{S} \). By hypothesis, \((xt) \subseteq \text{bd } S \), so let \( z \in (xt) \subseteq \text{S} \). Since \( S \) is starshaped relative to \( x \), for each \( m \in (xt) \subseteq \text{S} \) we have \((xz)_m \subseteq \text{S} \). Then, since we are denying the lemma, we may not find a sequence \( \{z_i\}_{i=1}^\infty \) such that \( z_i \in (x_m)_i \subseteq \text{S} \) for each \( i \), and \( ||z_i - x|| < \epsilon \) for \( i = 1 \). Then \( c \subseteq \text{int } S \) since \( x \) is a seeing point. Now let \( \{r_i\}_{i=1}^\infty \) be a sequence of boundary points converging to \( q \), such that \( r_i \neq q, r_i \in (tq) \) for each \( i \). Then \( \{r_i\}_{i=1}^\infty \) must approach \( q \) from the open half space generated by the line containing \([xt]\), opposite from the open half space containing \( c \). Choose \( \delta > 0 \), so that \( E = B(q, \delta) \subseteq \text{S} \) is convex. Then, clearly we have \( \text{int } (E \subseteq \text{S}) \neq \emptyset \). Since \( \{r_i\}_{i=1}^\infty \) converges to \( q \), there exists an integer \( k \), such that if \( i \geq k \), we have \( r_i \in E \). Let \( x_i \in \text{int } (E \subseteq \text{S}) \) be such that \( q \in (r_i x_i) \subseteq \text{S} \). This contradicts that \( q \) is a boundary point.

Returning to Case 2, let \( f \) be the homeomorphism mapping the closed unit disc onto \( S \). Then \( \text{bd } S \) is homeomorphic to the unit circle. Let \( f(x) = x \). Let \( y \neq x \) be on the unit circle. Then \( x \) and \( y \) naturally divide the unit circle into two nonempty, relatively open, disjoint, connected subsets, say \( B^1_{xy} \) and \( B^2_{xy} \). Let \( \{x_i\}_{i=1}^\infty \) and \( \{x_i^2\}_{i=1}^\infty \) be sequences of points in \( \text{bd } S \) such that the inverse images of \( x_i \) and \( x_i^2 \) under \( f \) are in \( B^1_{xy} \) and \( B^2_{xy} \), respectively, for each \( i \), and both sequences converge to \( x \). Now define the set \( S_i \) for each \( i \) to be that subset of \( S \) whose boundary is given by \( [x_i x] \cup [x_i x_i^2] \cup f(B_i) \) where \( B_i \) is \( B^1_{x_i x_i} \) if \( x \in E B^1_{x_i x_i} \) and \( B_i \) is \( B^2_{x_i x_i} \) if \( x \in E B^2_{x_i x_i} \). Now since \( S \) is
simply connected, $S_i \subset S$, and $S_i$ for each $i$ is well defined by Lemma 1. Then $\{S_i\}_{i=1}^{\infty}$ converges to $S$ in the Hausdorff metric. Since for each $i$, $x_i^1, x_i^2 \in C(S)$, we have $x_i^1, x_i^2 \in C(S_i)$. Then, the only possible point of local nonconvexity of $S_i$ is $x$. For some integer $k$, we must have for each $i \geq k$, that $x \in L(S_i)$ for otherwise $S$ would be convex. Now by Case 1 each $S_i$, $i \geq k$, is a union of two compact convex sets with mutually disjoint interiors. Thus, by Theorem 5, $S$ is a union of two compact convex sets with mutually disjoint interiors.

Case 3. Suppose there exists $z \in \text{bd } S$ such that $z \neq x$, $[xz] \subset \text{bd } S$ and there does not exist $z_1 \in \text{bd } S$, $z_1 \neq z$, such that $[xz] \subset [xz_1] \subset \text{bd } S$. Further, suppose for each $y \in \text{bd } S \sim [xz]$ that $[xy] \subset [xz]$. Then, following a similar proof as in Lemma 1 we show that if $y \in \text{bd } S \sim [xz]$, then $(xy) \subset \text{int } S$. Finally, as in Case 2, we construct a sequence of compact sets $\{S_i\}_{i=1}^{\infty}$ such that $|L(S_i)| = 1$ for each $i$ beyond some $k$ and $\{S_i\}_{i=1}^{\infty}$ converges to $S$ in the Hausdorff metric, and then apply Theorem 5.

**Proof of Theorem 1.** We begin with a lemma.

**Lemma 2.** Let $S$ be a two cell such that $|L(S)|$ is finite. Let $[xy]$ be a crosscut of $S$ such that $x, y \in C(S)$. Then there exists a convex subset $A_{xy}$ of $S$ such that $(xy) \subset A_{xy}$.

**Proof.** Choose $\varepsilon > 0$ so that $B(x, \varepsilon) \cap S$ is convex, $B(y, \varepsilon) \cap S$ is convex, $\text{cl } B(x, \varepsilon) \cap L(S) = \emptyset$ and $\text{cl } B(y, \varepsilon) \cap L(S) = \emptyset$. For each $z \in (xy)$, choose $\varepsilon_z > 0$ so that $\text{cl } B(z, \varepsilon_z) \subset S$. Since $[xy]$ is compact select a finite subcover $\{B(x, \varepsilon/2), B(y, \varepsilon/2), B(z_1, \varepsilon_{z_1}/2), \ldots , B(z_n, \varepsilon_{z_n}/2)\}$ of the cover $\{B(x, \varepsilon/2), B(y, \varepsilon/2), B(z, \varepsilon/2)\} \cup (xy)$. Choose $\delta > 0$ so that $\delta < \min \{\varepsilon/2, \varepsilon_{z_1}/2, \ldots , \varepsilon_{z_n}/2\}$. Let $B$ be the closure of the $\delta/\sqrt{2}$ ball about the set $[xy]$. Then $B \cap S$ is clearly connected, closed and $B \cap S \cap L(S) = \emptyset$. Then $B \cap S$ is convex by Tietze's Theorem and this is the required set $A_{xy}$.

Now suppose Theorem 1 is false. Let $r, s \in L(S)$ such that $C_r \cap L(S) = \emptyset$ or $C_s \cap L(S) = \emptyset$, which is possible since $|L(S)|$ is finite. Let us suppose that $C_r \cap L(S) = \emptyset$. Let $\mathcal{C} = \{[xy] | [xy] \text{ is a crosscut and } x, y \in C(S)\}$. Since $|L(S)|$ is finite, every interior point of $S$ is contained in an element of $\mathcal{C}$, so int $S \subset \cup \mathcal{C}$. Since we are assuming that Theorem 1 is false, if $[xy] \in \mathcal{C}$, then $x, y \in C_0^r$ or $x, y \in C_0^s$. Consider any two crosscuts in $\mathcal{C}$, say $[x_1y_1]$ and $[x_2y_2]$ such that $x_1, y_1 \in C_0^r$ and $x_2, y_2 \in C_0^s$. These crosscuts can not intersect. To see this suppose $z \in (x_1y_1) \cap (x_2y_2)$. Then the set $R$ whose boundary is given by $[x_1y_1]$ and the portion $B$ of bd $S$ in $C_0^r$ between $x_1$ and $y_1$, is convex by Tietze's Theorem. Since $B \cap [x_2y_2] = \emptyset$, this forces $x_2$ or $y_2$
to be an interior point of $R$, and hence an interior point of $S$, a contradiction.

To continue, the interior of $S$ is connected since it is the homeomorphic image of a connected set. Let $[xy] \in C$. Let $A_{xy}$ be as in Lemma 2. Then letting $\alpha_1 = \{ \text{int } A_{xy} \mid [xy] \in C, x, y \in C_i \}$ and $\alpha_2 = \{ \text{int } A_{xy} \mid [xy] \in C, x, y \in C_i \}$, we have $\text{int } S \subseteq \bigcup \alpha_1 \cup \bigcup \alpha_2$ since $U \subseteq \alpha_1 \cup \alpha_2$. Since $\text{int } S$ is connected, we must have $\bigcup \alpha_1 \cap \bigcup \alpha_2 \neq \emptyset$, which says that there exists $[xy_1]$ and $[xy_2]$ in $C$ with $x_1, y_1 \in C_1, x_2, y_2 \in C_2$, and $\text{int } A_{xy_1} \cap \text{int } A_{xy_2} \neq \emptyset$. Let $z \in \text{int } A_{xy_1} \cap \text{int } A_{xy_2}$. Then $z$ lies in a crosscut having endpoints in $C_1$ and in a crosscut having endpoints in $C_2$, which says these crosscuts intersect, which is a contradiction by the last paragraph. Thus Theorem 1 holds.

Proof of Theorem 2. We know Theorem 2 holds when $n = 0$, and $n = 1$ by Tietze’s Theorem and Theorem 6, respectively. Thus, assume the theorem holds for $0 \leq k \leq n - 1$ and we will show the result holds for $n$. We begin by considering cases. Let $x \in L(S)$.

Case 1. Suppose there exist $z, g \in \partial S$ such that $z \neq x, g \neq x, \ [zx] \cup [xg] \subseteq \partial S$ and $[zx] \cap [xg] = \{ x \}$. Then choose a crosscut $[xy]$ where $y \in C(S)$ such that $x \in C(D_{xy}) \cap C(D_{xy})$. Since $y \in C(S), y \in C(D_{xy}) \cap C(D_{xy})$. Now suppose $d = \left| L(D_{xy}) \right|$, then $d \leq n - 1$ and by hypothesis $D_{xy} = \bigcup_{i=1}^{d+1} C_i$ where for each $i, 1 \leq i \leq d + 1, C_i$ is compact and convex and for $1 \leq i, j \leq d + 1, i \neq j$, $\text{int } C_i \cap \text{int } C_j = \emptyset$. Since $\left| L(D_{xy}) \right| = d$ and since $x \in C(D_{xy}) \cap C(D_{xy})$, we have $\left| L(D_{xy}) \right| = n - d - 1$. Thus, by hypothesis $D_{xy} = \bigcup_{i=1}^{d+1} B_i$ where for each $i, 0 \leq i \leq n - d, B_i$ is compact, convex, and for $i \neq j, 1 \leq i, j \leq n - d$, $\text{int } B_i \cap \text{int } B_j = \emptyset$. Thus $S$ is a union of $(d + 1) + (n - d) = n + 1$ compact convex sets with mutually disjoint interiors.

Case 2. Suppose for each $y \in \partial S, y \neq x$, that $[xy] \subseteq \partial S$. We shall need the following lemma.

Lemma 3. Let $S$ be a two cell such that $\left| L(S) \right| = n \geq 1$ satisfying Case 2. Let $\{ x_i \}_{i=1}^{\infty}$ be a sequence in $\partial S$ with $x_i \rightarrow x, x \in L(S)$ and $x_i \neq x$ for each $i$. Then there exists an integer $k$ such that for each $i \geq k, (xx_i) \subseteq \text{int } S$.

Proof. We proceed by induction. We know by Lemma 1 the lemma holds when $\left| L(S) \right| = 1$. So suppose the lemma holds for $1 \leq k \leq n - 1$. By Theorem 1, there exists a crosscut $[rs]$ of $S$ such that $r, s \in C(S)$ and $\left| L(D_{rs}) \right| \leq n - 1$ for $i = 1, 2$. Suppose $x \in L(D_{rs})$. Then since there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S \subseteq D_{rs}$, the result holds. The same argument applies if $x \in D_{rs}$.

To complete Case 2, we argue the same way as in Case 2 of Theorem 6. We construct a sequence of sets $\{ S_i \}_{i=1}^{\infty}$ such that $\{ S_i \}_{i=1}^{\infty}$ converges to $S$ in the Hausdorff metric and using Case 1 of this theorem.
we show each $S_i$ is representable as $n+1$ compact convex sets with mutually disjoint interiors and then apply Theorem 5. This completes Case 2.

**Case 3.** Suppose there exists $x \in L(S)$ and $z \in \text{bd } S$, $z \neq x$ such that $[xz] \subseteq \text{bd } S$ and there does not exist $z_1 \in \text{bd } S$, $z_1 \neq z$ such that $[xz] \subseteq [xz_1] \subseteq \text{bd } S$. Further suppose for each $y \in \text{bd } S \sim [xz]$, that $[x, y] \subseteq \text{bd } S$. Then using a similar proof as in Lemma 3, we prove that if $\{x_i\}_{i=1}^\infty$ is a sequence in $\text{bd } S \sim [xz]$ converging to $x$, then there exists an integer $k$, such that for each $i \geq k$, $(x_i, x) \subseteq \text{int } S$ and use the same construction as in Case 2 to get the theorem. This completes the proof of Theorem 2.

**III. Consequences of Theorem 2.** As immediate corollaries of Theorem 2 we have

**Corollary 1.** Let $S$ be a two cell such that $|L(S)| = n$. Then $S$ is an $L_{n+1}$ set.

**Corollary 2.** Let $S$ be a two cell such that $|L(S)| = n$. Then for every line $L$, $L \cap S$ is a union of at most $n+1$ closed line segments.

Corollary 2 is a generalization of the familiar fact that the intersection of a line and a compact convex set is a point or a closed line segment.

Examples are easily constructable to show the number $n+1$ is best in Theorem 2.

**References**

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