SOLVABLE AUTOMORPHISM GROUPS AND
AN UPPER BOUND FOR \( |A(G)| \)

J. R. WEAVER

Abstract. The objective of this work is to find a subgroup \( H \) of a finite group \( G \) which will give information about the order of the automorphism group of \( G \) and the structure of the automorphism group of \( G \). An upper bound is found for the order of the automorphism group and conditions are given which insure that the automorphism group is solvable. Some information is given about a normal subgroup of a particular subgroup of the automorphism group. In this paper all groups are assumed to be finite.

1.0. Notations. \( H \trianglelefteq G, H \triangledown G \), shall mean respectively, that \( H \) is a subgroup, a normal subgroup of a group \( G \).

\( A(G) \) shall mean the automorphism group of \( G \).

\( \langle S \rangle \) = subgroup generated by the subset \( S \) in \( G \).

\( N_G(H) \) = the normalizer of the subset \( H \) in the group \( G \).

\( H^x = x^{-1} H x \) for \( x \) in \( G \).

\( [G : H] \) = index of the subgroup \( H \) in \( G \).

\( g^\alpha \) is the image of the element \( g \) under the mapping \( \alpha \).

\( H^\alpha \) is the image of the subset under the mapping \( \alpha \).

\( F(G ; H) = \{ \alpha \in A(G) \mid h^\alpha = h \text{ for each } h \text{ in } H \} \).

\( |G| \) is the number of elements in the group \( G \).

\( I(G) \) shall mean the inner automorphism group of the group \( G \).

1.1. Basic properties of \( H \)-automorphisms. Hans Liebeck introduced the notation for \( H \)-automorphism which is adopted.

Definition 1.1.1. Let \( H \trianglelefteq G \). An automorphism \( \alpha \) of \( G \) is called an \( H \)-automorphism if \( g^{-1} g^\alpha \in H \) for each \( g \) in \( G \). Denote the set of all \( H \)-automorphisms by \( A(G ; H) \).

If \( H \trianglelefteq G \), then \( A(G ; H) \trianglelefteq A(G) \). Conversely, any subgroup \( T \) of \( A(G) \) can be regarded as a group of \( H \)-automorphisms for various choices of \( H \). Since \( A(G ; G) = A(G) \), one can always choose \( H = G \).

Let \( T \trianglelefteq A(G) \) for a group \( G \). \( K(G ; T) \) is used to represent \( \langle g^{-1} g^\alpha \mid \alpha \in T, g \in G \rangle \). \( K(G ; T) \) is called the multiplier subgroup of \( T \). Liebeck \([3]\) has shown that \( T \) may be a proper subgroup of the group of all \( K(G ; T) \)-automorphisms. One easily shows that \( K(G ; T) \Delta G \).
The set of \( \alpha \in A(G) \) such that \( H^\alpha = H \) for \( H \) a subgroup of \( G \) is denoted by \( \Gamma(G; H) \). \( \Gamma(G; H) \) is clearly a subgroup of \( A(G) \) which contains \( A(G; H) \). This brings us to

**Proposition 1.1.2.** If \( H \trianglelefteq G \), then \( A(G; H) \trianglelefteq \Gamma(G; H) \).

**Proof.** Let \( \alpha \in A(G; H) \) and let \( \gamma \in \Gamma(G; H) \). If \( g \in G \) then \( g^{-1} \gamma g^{-1} \gamma = h \gamma \in H \) for some \( h \) in \( H \). Therefore \( A(G; H) \trianglelefteq \Gamma(G; H) \).

Since \( A(G; H) \trianglelefteq \Gamma(G; H) \), an obvious question to ask is, "What relationship exists between \( \Gamma(G; H) \), \( A(G; H) \), and \( K(G; A(G; H)) \)?" The following will give some information in this direction.

**Lemma 1.1.3.** If \( H \trianglelefteq G \), then \( K(G; A(G; H)) \) is admissible with respect to \( \Gamma(G; H) \).

**Proof.** If \( \gamma \in \Gamma(G; H) \) and \( g^{-1} \gamma \) is a generator of \( K(G; A(G; H)) \), then

\[
(g^{-1} \gamma) \gamma = (g^{-1} \gamma)^{-1} (g^{-1} \gamma) \gamma \in K(G; A(G; H)).
\]

Therefore \( K(G; A(G; H)) \) is admissible with respect to \( \Gamma(G; H) \).

**Theorem 1.1.4.** If \( H \trianglelefteq G \), then there exists a normal subgroup \( M \) of \( G \) such that \( K(G; A(G; H)) \subseteq M \trianglelefteq H \), \( \Gamma(G; H)/A(G; H) \) is isomorphic to a subgroup of \( A(G/M) \), and \( M \) is admissible with respect to \( \Gamma(G; H) \). \( M \) is also maximal with respect to being normal in \( G \), being a subgroup of \( H \), and being admissible with respect to \( \Gamma(G; H) \).

**Proof.** \( K(G; A(G; H)) \) is normal in \( G \), is a subgroup of \( H \), and is admissible with respect to \( \Gamma(G; H) \). Let \( M \) be a subgroup of \( G \) maximal with respect to these three properties.

Now define \( \phi \) mapping \( \Gamma(G; H) \) into \( A(G/M) \) by \( (g^\sigma)^{(\gamma)} \phi = (g^\gamma)^\phi \) where \( \sigma \) is the natural homomorphism of \( G \) onto \( G/M \) and \( \gamma \in \Gamma(G; H) \). If \( g^\sigma \) and \( b^\sigma \) are two elements of \( G/M \), then \( (g^\sigma b^\sigma)^{(\gamma)} \phi = (g^\sigma)^{(\gamma)} \phi (b^\sigma)^{(\gamma)} \phi \).

It then follows that \( (\gamma) \phi \) is an automorphism of \( G/M \) for each \( \gamma \) in \( \Gamma(G; H) \).

For \( \gamma \) and \( B \) in \( \Gamma(G; H) \), \( (g^\sigma)^{(\gamma B)} \phi = (g^\sigma)^{(\gamma \phi(B) \phi)} \). It follows from this that \( \phi \) is a homomorphism of \( \Gamma(G; H) \) into \( A(G/M) \).

The kernel of \( \phi \) is \( A(G; H) \). Therefore \( \Gamma(G; H)/A(G; H) \) is isomorphic to a subgroup of \( A(G/M) \).

If \( H \trianglelefteq G \), then \( \Gamma(G; H)^{\alpha} = \Gamma(G; H^\alpha) \) and \( A(G; H)^{\alpha} = A(G; H^\alpha) \) for \( \alpha \) in \( A(G) \).

1.2 \( A \)-invariant and \( A \)-closed subgroups. Helmut W. Wielandt [4] calls a subgroup \( H \) of \( G \) invariant in \( G \) if and only if for each automorphism \( \alpha \) in \( A(G) \) there exists a \( g \) in \( G \) such that \( (H^\alpha)^g = H \).

This definition is generalized to give
Definition 1.2.1. A subgroup $H$ of $G$ is $A$-intravariant in $G$ for $A$ a subgroup of $A(G)$ if and only if for each automorphism $\alpha \in A$ there exists a $g$ in $G$ such that $(\alpha g)^* = H$.

Some of the better known $A(G)$-intravariant subgroups are Sylow subgroups and characteristic subgroups of any group $G$; and Hall subgroups, Carter subgroups, and system normalizers of solvable groups.

Proposition 1.2.2. If $H \leq G$, then $H$ is $A$-intravariant in $G$ for $A$ a subgroup of $A(G)$ if and only if $A \leq \Gamma(G; H)I(G)$.

Proof. Since $I(G) \Delta A(G)$, $\Gamma(G; H)I(G)$ is a well-defined subgroup of $A(G)$. If $H$ is $A$-intravariant then for each $\alpha \in A$ there exist a $g$ in $G$ such that $(\alpha g)^* = H$. Therefore $\alpha \in \Gamma(G; H)I(G)$ and $A \leq \Gamma(G; H)I(G)$

If $A \leq \Gamma(G; H)I(G)$ then each element in $A$ may be represented as the product of an element from $\Gamma(G; H)$ and an element from $I(G)$. Therefore $H$ is $A$-intravariant.

Upon considering the subgroup $\Gamma(G; H) \cap I(G)$ for $H \leq G$, it is easily shown that $\Gamma(G; H) \cap I(G) \cong \mathcal{N}_G(H)/Z(G)$. This leads to

Proposition 1.2.3. Let $H \leq G$ and assume that $I(G) \leq A \leq A(G)$. Then $H$ is $A$-intravariant if and only if $[G: \mathcal{N}_G(H)] = [A : A \cap \Gamma(G; H)]$.

Proof. Since $A \leq \Gamma(G; H)I(G)$,

$$A = A \cap \Gamma(G; H)I(G) = I(G)(A \cap \Gamma(G; H)).$$

Hence

$$|A| = |I(G)| \cdot |A \cap \Gamma(G; H)| / |\Gamma(G; H) \cap I(G)|.$$

Therefore $[A : A \cap \Gamma(G; H)] = [G: \mathcal{N}_G(H)]$.

Conversely; suppose that $[G: \mathcal{N}_G(H)] = [A : A \cap \Gamma(G; H)]$. Since $I(G) \leq A$, $I(G)(A \cap \Gamma(G; H)) \leq A$. Note that $|A| = |I(G)(A \cap \Gamma(G; H))|$. Hence $A = I(G)(A \cap \Gamma(G; H)) = A \cap \Gamma(G; H)I(G)$. Therefore $A \leq \Gamma(G; H)I(G)$ and $H$ is $A$-intravariant.

It is easy to show that if $H$ is $A$-intravariant in $G$, then $\mathcal{N}_G(H)$ is $A$-intravariant in $G$ for $A$ a subgroup of $A(G)$.

In 1937 Philip Hall [2] showed that if $G$ is a solvable group, then the Sylow system $S$ is $A(G)$-intravariant. Then he showed that $A(G) = I(G)\Gamma(G; S)$. With this he was able to arrive at an upper bound for the order of $A(G)$. Later on an upper bound for $|A(G)|$ will be found.

1.3. $A$-closed subgroups. Recall that $H$ is weakly closed in the subgroup $K$ if $H^g \leq K \leq G$ implies that $H^g = H$ for $g$ in $G$.

Definition 1.3.1. A subgroup $H$ of a group $G$ is $A$-closed in $G$ for $A$ a subgroup of $A(G)$ if:
(i) $H$ is $A$-intravariant in $G$, and,
(ii) $H$ is weakly closed in $N_G(H)$.

**Proposition 1.3.2.** If $H \leq G$, then $A(G; H) \leq \Gamma(G; H^e) \leq A(G; N_G(H))$.

**Proof.** Assume that $a \in A(G; H)$. It then follows that $(h^a)^{-1}(h^a)a \in K(G; A(G; H)) A(G)$. Therefore $K(G; A(G; H)) \leq H^e$ and $(h^a)^e \in H^e$.

Suppose that $a \in \Gamma(G; A(G; H))$. If $g \in G$ and $h \in H$, then $h^e = (x^e)^a$ for some $x$ in $H$. On the other hand, $(x^e)^a = (x^a)^{(x^a)^{-1}} = h^e$. Hence $x^a = h^e x^a$ is an element of $H$ and $g^{-1}g^a \in N_G(H)$. Therefore $a \in A(G; N_G(H))$. It is useful to know when $\langle a \rangle \leq \Gamma(G; H^e) = A(G; N_G(H))$. This brings us to

**Proposition 1.3.3.** If $H$ is a $\Gamma(G; N_G(H))$-closed subgroup of $G$, then $A(G; N_G(H)) = \Gamma(G; H^e)$.

**Proof.** By 1.3.2, $\Gamma(G; H^e) = A(G; N_G(H))$. It is easily shown that $\Gamma(G; H^e) = \Gamma(G; N_G(H^e))$ for each $g$ in $G$. Therefore $A(G; N_G(H)) = \Gamma(G; N_G(H))$. The converse of 1.3.3. is not true. Let $H$ be the subgroup of the alternating group of degree 5 which is equal to $\langle (1, 2), (3, 4) \rangle$. $N_G(H) = \langle (1, 2), (3, 4), (1, 3)(2, 4) \rangle$ and it can be shown that $H$ is not weakly closed in $N_G(H)$.

1.4. $A(G)$-closed and $A(G)$-intravariant subgroups. In this section the existence of an $A(G)$-closed subgroup $H$ in the group $G$ will be used to establish the structure of $A(G)$.

**Lemma 1.4.1.** If $H$ is an $A(G)$-intravariant subgroup of $G$, then $A(G)$ is solvable if and only if $\Gamma(G; H) / \Gamma(G; H) \cap I(G)$ and $G$ are solvable.

**Proof.** If $A(G)$ is solvable then it is obvious that $\Gamma(G; H) / \Gamma(G; H) \cap I(G)$ and $G$ are solvable.

Conversely, suppose that $G$ and $\Gamma(G; H) / \Gamma(G; H) \cap I(G)$ are solvable. By 1.2.2 $A(G) = \Gamma(G; H)I(G)$. Hence

$$A(G) / I(G) \cong \Gamma(G; H) / \Gamma(G; H) \cap I(G).$$

Since $G$ is solvable, $I(G)$ is solvable and $A(G)$ is solvable.

**Theorem 1.4.2.** Suppose that $H$ is $A(G)$-closed in the solvable group $G$ with $\Gamma(G; H)$ a maximal subgroup of $A(G)$. If $\Gamma(G; H) / \Gamma(G; H) \cap I(G)$ is solvable with $A(G; N_G(H)) \cong \langle 1 \rangle$, then $A(G)$ is solvable and there exists a minimal normal subgroup $T$ of $A(G)$ and $A(G)$ satisfies the following:

(i) $C_{A(G)}(T) = T$;

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(ii) $T$ is the minimal normal subgroup of $A(G)$;
(iii) $A(G) = \Gamma(G; H)T$;
(iv) $T \cap \Gamma(G; H) \cong \langle 1 \rangle$;
(v) $|T| = |A(G) : \Gamma(G; H)|$;
(vi) if $G_0/T$ is a minimal normal subgroup of $A(G)/T$ and $A_0 = \Gamma(G; H) \cap G_0$, then $\Gamma(G; H) = N_{A(G)}(A_0)$, $|G_0| = |T| |A_0|$, $|T| = p^a$, $|A_0| = q^b$ and $p \neq q$ for primes $p$ and $q$ and nonnegative integers $a$ and $b$.

**Proof.** By 1.4.1, $A(G)$ is a solvable group. Hence $A(G)$ has a minimal normal subgroup $T$.

Note that $A(G; A_0) = \Gamma(G; H)$, and $A(G; A_0) = \Gamma(G; H)$. It follows from [4] that $A(G) = \Gamma(G; H)T$ where $T$ is the minimal normal subgroup of $G$. Then $C_{A(G)}(T) = T$, $\Gamma(G; H) \cap T \cong \langle 1 \rangle$, $[A(G) : \Gamma(G; H)] = T$, and condition (vi) is satisfied.

The condition that $\Gamma(G; H)$ be a maximal subgroup of $A(G)$ could be replaced by $[G : N_{\sigma}(H)]$ is a prime number. Then $|T| = [G : N_{\sigma}(H)]$.

Continuing in the search for groups whose automorphism groups are solvable we arrive at

**Proposition 1.4.3.** Let each subgroup in a composition series of the subgroup $H$ of the solvable group $G$ be $\Gamma(G; H)$-closed for $H$ an $A(G)$-intravariant subgroup of $G$. If $F(G; H)$ is solvable then $A(G)$ is solvable.

**Proof.** By 1.4.1, $A(G)$ is solvable if $\Gamma(G; H)/T(G; H) \cap I(G)$ is solvable. Note that $F(G; H) = \Gamma(G; H)$.

Let $\langle 1 \rangle = N_0 \leq N_{r-1} \leq \cdots \leq N_r = H$ be the composition series of $H$ mentioned in the hypothesis. Using induction it is easy to show that each $N_i \triangle H$ for $i = 0, 1, \cdots, r$. If $\alpha \in \Gamma(G; H)$, then $N_i^\alpha = N_i \leq H$ for some $g$ in $G$. Since $N_i^g \leq H \leq N_{\sigma}(H)$ we have that $N_i^g = N_i$. Therefore $\Gamma(G; H)$ leaves each subgroup in the series fixed. Note that $\Gamma(G; H)/F(G; H)$ is isomorphic to a subgroup of $A(H)$. By Gaschütz [1], $\Gamma(G; H)/F(G; H)$ is solvable, $\Gamma(G; H)$ is solvable and the proof is complete.

**1.5. An upper bound for the order of $A(G)$.** An upper bound for $|A(G)|$ will be derived by using the concept of $H$ being $A(G)$-intravariant in $G$.

**Theorem 1.5.1.** Let $H$ be an $A(G)$-intravariant subgroup of $G$ with $F(G; K_1) \leq A(G; H)$ where $K_1 = K(G; A(G; H))$. Then $|A(G)|$ divides $|G : N_{\sigma}(H)| |A(K_1)| |F(G; K_1)|$. The prime divisors of $|F(G; K_1)|$ are prime divisors of $|K_1|$.

**Proof.** If $p$ is a prime dividing $|F(G; K_1)|$ then there exists an $\alpha$ in $F(G; K_1)$ such that $|\alpha| = p$. Since $\alpha$ is not the identity, there exists a $g$
in $G$ such that $g^{-1}g^a = k \in K_1$, $k \neq 1$. Hence $g^a g = g k^p g = g$. Therefore $k^p = 1$ and $p$ divides $|K_1|$.

By 1.1.3, $K_1$ is admissible with respect to $\Gamma(G; H)$. Define $\phi$ mapping $\Gamma(G; H)$ into $A(K_1)$ by $(k)^{\gamma} = k^\gamma$ for each $k$ in $K_1$. $\phi$ is a homomorphism of $\Gamma(G; H)$ into $A(K_1)$ and the kernel of $\phi$ is $F(G; K_1)$. Therefore $\Gamma(G; H)/F(G; K_1)$ is isomorphic to a subgroup of $A(K_1)$.

It follows from 1.2.3 that $[A(G):\Gamma(G; H)] = [G:N_G(H)]$. Since $|\Gamma(G; H)/F(G; K_1)|$ divides $|A(K_1)|$, $|A(G)|$ divides $[G:N_G(H)] |A(K_1)| |F(G; K_1)|$.

We are now in a position to prove

**Corollary 1.5.2.** Let $H$ be a characteristic subgroup of $G$ such that $F(G; K_1) \subseteq A(G; H)$ where $K_1 = K(G; A(G; H))$ is a cyclic $p$-group of $G$, then $|A(G)|$ divides $p^m(p - 1)$ for some nonnegative integer $m$.

**Proof.** If $|K_1| = 2^t$ then $|A(G)|$ divides some power of 2.

Assume that the order of $K_1$ is equal to $p^t$ for an odd prime $p$. Then $|A(K_1)| = p^{t-1}(p - 1)$ and $|A(G)|$ divides $p^m(p - 1)$ for some nonnegative integer $m$.

**Corollary 1.5.3.** Let $H$ be a characteristic subgroup of $G$ such that $K_1 = K(G; A(G; H))$ is a cyclic $p$-group with $p$ a prime of the form $2^m + 1$ for some integer $m$. If $F(G; K_1) \subseteq A(G; H)$ then $|A(G)|$ divides $p^{2m}$, $A(G)$ is solvable, and $G$ is solvable.

**Proof.** By 1.5.2, $|A(G)|$ divides $p^{2m}$ for some integer $n$. Hence $A(G)$ and $G$ are solvable.

**Corollary 1.5.4.** If $H$ is a normal Sylow 2-subgroup of $G$ such that $F(G; K_1) \subseteq A(G; H)$ for $K_1 = K(G; A(G; H))$ cyclic, then $G = H$.

**Proof.** By 1.5.2, $|A(G)| = 2^n$ for some integer $n$. Hence $A(G)$ is a nilpotent group and $G$ is nilpotent.

Assume that there exists a prime $p \neq 2$ which divides $|G|$. Since $G$ is a nilpotent group, $A(G)$, is a direct product of the automorphism groups of the Sylow subgroups of $G$.

Let $\alpha \in A(H_p)$ where $H_p$ is the Sylow $p$-subgroup of $G$. Note that $\alpha$ can be thought of as an element in $A(G)$ that acts as the identity on $H$ and is an element of $A(G; H)$. If $g \in H_p$ then $g^{-1}g^a \in H \cap H_p = \langle 1 \rangle$. Therefore $\alpha$ is the identity on $H_p$ and $A(H_p) = \langle 1 \rangle$. Since $2 < p$, we have a contradiction. Therefore $H = G$.

The upper bound for $|A(G)|$ found in 1.5.1 is actually attained if $G$ is the symmetric group of degree 3 and $H$ is the alternating group of degree 3.
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MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823

THE UNIVERSITY OF WEST FLORIDA, PENSACOLA, FLORIDA 32504