

PRIME 3-MANIFOLDS AND THE DOUBLING OPERATION

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ABSTRACT. Examples are given to show that, in general, one may not factor a compact 3-manifold M with nonvacuous boundary into primes relative to the multi-disk sum (a boundary pasting operation) by factoring the double of M into primes relative to the connected sum. Necessary and sufficient conditions for the double of a compact 3-manifold with nonvacuous boundary to be prime relative to the connected sum are established.

1. **Introduction.** Any 3-manifold considered here is triangulated, oriented, connected, and compact. Any map considered is piecewise linear. A 3-sphere or a 3-cell will sometimes be called a trivial 3-manifold.

Milnor [4] has shown that each 3-manifold with vacuous boundary has an essentially unique factorization into $\#$ -primes, and the author [1] has shown that each 3-manifold with connected, nonvacuous boundary has an essentially unique factorization with Δ -primes. This paper gives the necessary and sufficient conditions for the double of a Δ -prime 3-manifold to be a $\#$ -prime 3-manifold. It also gives some examples of 3-manifolds with radically different topological properties whose doubles are homeomorphic.

Definitions of certain operations on 3-manifolds and the respective notions of primeness which follow from them are given in §2 along with precise statements of the factorization theorems of Milnor and the author. Theorem 1 of §3 provides a sufficient condition for pasting together two m -prime 3-manifolds (which are defined in §2) so that one obtains a $\#$ -prime 3-manifold. The necessary and sufficient conditions for the double of a Δ -prime 3-manifold to be a $\#$ -prime 3-manifold are derived in §4 from the work of §3. The examples concerning the doubling operation are given in §4.

In what follows, an imbedding of a manifold M in a manifold N is called *proper* if $\text{bd}(M) \subset \text{bd}(N)$ and $\text{int}(M) \subset \text{int}(N)$.

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2. **Prime factorization theorems.** The *connected sum* $M\#M'$ of two disjoint 3-manifolds is obtained by removing from each the interior

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of a 3-cell and then pasting the resulting boundary components together with an orientation reversing homeomorphism. Up to homeomorphism, the operation of connected sum is well defined, associative, and commutative. One says that a 3-manifold P is $\#$ -prime if P is not a 3-sphere, and whenever P is homeomorphic to a connected sum $M\#M'$, either M or M' is a 3-sphere. Milnor [4] has shown that every nontrivial 3-manifold is homeomorphic to a connected sum $P_1\#\cdots\#P_n$ of $\#$ -prime 3-manifolds, which are uniquely determined up to order and homeomorphism. Of course, if a 3-manifold has nonvacuous boundary then some of its $\#$ -prime summands have nonvacuous boundary.

If M and M' are two disjoint 3-manifolds with connected nonvacuous boundary, one forms their *disk sum* $M\Delta M'$ by pasting a 2-cell on $\text{bd}(M)$ to a 2-cell on $\text{bd}(M')$ with an orientation reversing homeomorphism. Up to homeomorphism, the operation of disk sum is well defined, associative, and commutative. One says that a 3-manifold P with connected nonvacuous boundary is Δ -prime if P is not a 3-cell and whenever P is homeomorphic to a disk sum $M\Delta M'$, either M or M' is a 3-cell. The author [1] has shown that every nontrivial 3-manifold with connected nonvacuous boundary is homeomorphic to a disk sum $Q_1\Delta\cdots\Delta Q_n$ of Δ -prime 3-manifolds, which are uniquely determined up to order and homeomorphism, the exact analogue of Milnor's result.

Let M be a 3-manifold with r boundary components and let M' be a 3-manifold with s boundary components ($r, s \geq 1$). For any $j \leq \min(r, s)$, one forms a *multi-disk sum* of M and M' by pasting j disks on $\text{bd}(M)$, no two of which lie on the same component of $\text{bd}(M)$, to j disks on $\text{bd}(M')$, no two of which lie on the same component of $\text{bd}(M')$, with an orientation reversing homeomorphism. For each j , there may be many distinct (up to homeomorphism) multi-disk sums of the 3-manifolds M and M' . A nontrivial 3-manifold P is called *m-prime* if whenever P is homeomorphic to a multi-disk sum of two 3-manifolds, one of the summands is a 3-cell. One observes that a Δ -prime 3-manifold is simply an *m-prime* 3-manifold with connected boundary. The author [2] has established decomposition results for 3-manifolds with several boundary components, which are only partially analogous to his results in [1].

3. Δ -irreducibility. A 3-manifold M is called *irreducible* if every imbedded 2-sphere in M bounds a 3-cell. Proof of the following proposition is given by Milnor [4]:

PROPOSITION 1. *Any 3-manifold which is irreducible but not #-prime is homeomorphic to the 3-sphere S^3 , and any 3-manifold which is #-prime but not irreducible is homeomorphic to $S^1 \times S^2$.*

A 3-manifold M with nonvacuous boundary is called Δ -irreducible if every properly imbedded disk in M separates M and if the closure of at least one component of the complement of the disk is a 3-cell.

PROPOSITION 2. *Any 3-manifold which is Δ -irreducible but not m -prime is a 3-cell, but there are infinitely many m -prime (in particular, infinitely many Δ -prime) 3-manifolds which are not Δ -irreducible.*

PROOF. Let B be a Δ -irreducible 3-manifold, so every properly imbedded disk in B separates B . Thus, if B is a multi-disk sum of the 3-manifolds M_1 and M_2 , then $M_1 \cap M_2$ is a single disk. The Δ -irreducibility of B further implies that either M_1 or M_2 is a 3-cell, which implies that B is either an m -prime 3-manifold or a 3-cell. The solid torus of genus one is the example of an m -prime 3-manifold which is not Δ -irreducible, that is the analogue of $S^1 \times S^2$. The closure of the complement in S^3 of the union of a finite family of inseparably linked solid tori of genus one is Δ -irreducible and m -prime, and, of course, there are infinitely many inseparable links of any multiplicity. Theorem 7 of [2] states that if P is any m -prime 3-manifold with more than one boundary component, and if Q is a 3-manifold obtained by identifying two disks on different components of $\text{bd}(P)$ with an orientation reversing homeomorphism, then Q is m -prime; but, obviously, Q is not Δ -irreducible. Thus, the inseparable links yield infinitely many examples of m -prime 3-manifolds which are not Δ -irreducible.

PROPOSITION 3. *Any 3-manifold which is m -prime but not #-prime is homeomorphic either to $S^2 \times [0, 1]$ or to the result of removing from some #-prime 3-manifold with vacuous boundary the interior of a 3-cell. There are infinitely many #-prime 3-manifolds with nonvacuous boundary which are not m -prime.*

PROOF. The first statement follows at once from Lemma 9 of [2]. Lemma 5 of [2] states that a multi-disk sum of irreducible 3-manifolds is an irreducible 3-manifold, which implies the second statement.

Following W. Haken (as in Waldhausen [7]), one defines a component of the boundary of a nontrivial 3-manifold M to be *incompressible in M* if the kernel of the induced map $\pi_1(S) \rightarrow \pi_1(M)$ is trivial

(note the Loop Theorem and Dehn's Lemma of Papakyriakopoulos [5] and [6]).

THEOREM 1. *Let P' and P'' be disjoint, irreducible nontrivial 3-manifolds. Let R' and R'' each be the union of a family of incompressible components of $\text{bd}(P')$ and $\text{bd}(P'')$, respectively. And let $f: R' \rightarrow R''$ be an orientation reversing homeomorphism. Then the 3-manifold P obtained from P' and P'' by identifying R' and R'' via the map f is irreducible.*

PROOF. Let R denote the image in P of R' and R'' , let S be a 2-sphere in P lying in general position with respect to R . The proof that S bounds a 3-cell in P is by an induction argument on the number of components of $S \cap R$. If this number is zero, then S bounds a 3-cell because it lies either in P' or in P'' , both of which are irreducible.

Otherwise $S \cap R$ contains a loop k which bounds a disk D on the sphere S such that $\text{int}(D) \cap R$ is void. For definiteness, one supposes that the disk D lies in P' . The loop k bounds a disk E on R because R is incompressible in P' . The 2-sphere $D \cup E$ bounds a 3-cell in P' because P' is irreducible. Therefore, there is a 2-sphere S^* in P which is isotopic to S such that $S^* \cap R$ has fewer components than $S \cap R$. By the induction hypothesis, S^* bounds a 3-cell. The existence of an isotopy $S \rightarrow S^*$ implies that S also bounds a 3-cell.

COROLLARY. *Let P' and P'' be disjoint, irreducible, Δ -irreducible, nontrivial 3-manifolds, and let $f: \text{bd}(P') \rightarrow \text{bd}(P'')$ be a homeomorphism. Then the 3-manifold P obtained by pasting P' to P'' via the map f is $\#$ -prime.*

EXAMPLE 1. Let P and Q be disjoint 3-manifolds, each obtained by taking the closure of the complement in a 3-sphere of a nontrivially knotted solid torus of genus one. The corollary to Theorem 1 implies that the result of pasting $\text{bd}(P)$ to $\text{bd}(Q)$ via any homeomorphism is a $\#$ -prime 3-manifold. One shows with a Mayer-Vietoris sequence that the result of pasting with a homeomorphism which takes the longitude of $\text{bd}(P)$ onto the meridian of $\text{bd}(Q)$ and the meridian of $\text{bd}(P)$ onto the longitude of $\text{bd}(Q)$ is a homology 3-sphere. W. Browder has suggested that it might be possible to construct in this manner infinitely many topologically distinct $\#$ -prime homology 3-spheres, which was accomplished by the author in [3].

4. The doubling operation. Let M be a 3-manifold with non-vacuous boundary. The *double* of M is obtained by identifying the boundary of M with the boundary of a homeomorphic copy of M

(disjoint from M) under the restriction to $\text{bd}(M)$ of a homeomorphism of M onto the copy or, equivalently, as $\text{bd}(M \times I)$.

Several times the author has been asked whether his decomposition results of [1] and [2] might be obtained by employing the doubling operation. The following three examples indicate some difficulties which arise in such an approach:

EXAMPLE 2. The solid torus of genus one and $S^2 \times [0, 1]$ are m -prime 3-manifolds which do not have the same number of boundary components, but their doubles are both homeomorphic to the $\#$ -prime 3-manifold $S^2 \times S^1$.

EXAMPLE 3. Let M be the closure of the complement in S^3 of three inseparably linked solid tori of genus one such that any autohomeomorphism of M takes each component of $\text{bd}(M)$ onto itself, and let M' be a homeomorphic copy of M which is disjoint from M . Choose two components of $\text{bd}(M)$ and form a 3-manifold P by pasting them together across a disk. Choose two components of $\text{bd}(M')$, one of which is not the image of either of the two components of $\text{bd}(M)$ chosen in the construction of P under any homeomorphism, and form a 3-manifold Q by pasting them together across a disk. One observes that P and Q are not homeomorphic and that both P and Q have two boundary components. Theorem 7 of [2] implies that both P and Q are m -prime. The double of P and the double of Q are both homeomorphic to the connected sum of the double of M and $S^2 \times S^1$.

EXAMPLE 4. Let P be homeomorphic to the Cartesian product of the circle S^1 with the complement in the torus T^2 of the interior of a disk. Let M be the disk sum of P with a solid torus of genus one, so that M is not Δ -prime. Let Q be obtained from $(T^2 \# T^2) \times [0, 1]$ by pasting a disk on $(T^2 \# T^2) \times \{0\}$ to a disk on $(T^2 \# T^2) \times \{1\}$ with an orientation reversing homeomorphism. Theorem 7 of [2] implies that Q is Δ -prime. One observes that the double of M and the double of Q are both homeomorphic to the connected sum of $(T^2 \# T^2) \times S^1$ and $S^2 \times S^1$.

THEOREM 2. *Let P be a m -prime 3-manifold which is not a solid torus or homeomorphic to $S^2 \times [0, 1]$. Then the double of P is $\#$ -prime if and only if P is irreducible and Δ -irreducible.*

PROOF. The corollary to Theorem 1 implies the sufficiency of irreducibility and Δ -irreducibility. Lemma 9 of [2] implies the necessity of irreducibility. To show the necessity of Δ -irreducibility, suppose that D is a properly imbedded disk in P which does not separate

P . The 3-manifold Q obtained by splitting P at D (in the sense of Waldhausen [7] and others) is nontrivial, and the double of P is homeomorphic to the connected sum of the double of Q and $S^2 \times S^1$.

It is even possible for two nonhomeomorphic Δ -irreducible 3-manifolds with connected boundary to have homeomorphic doubles, as shown by an unpublished example of C. Feustel.

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