FACTORIZATION OF DIFFERENTIAL OPERATORS

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Abstract. A necessary and sufficient condition for a differential operator of order \( n \) to be factorable into a product of operators of orders \( n-r \) and \( r \), for any \( 0 < r < n \), is given.

Consider differential operators of the form

\[
Ly = s_0 y^{(n)} + s_1 y^{(n-1)} + \cdots + s_n y 
\]

with \( s_i \) continuous for \( i = 0, \ldots, n \) and \( s_0(t) \neq 0 \).

Theorem. Suppose \( L \) is an operator of order \( n \) of type (*) and \( 0 < r < n \). Then there exist operators \( P \) and \( Q \) both of type (*) of orders \( n-r \) and \( r \) respectively such that \( L = PQ \) on some interval \( I \) if and only if there exist \( r \) linearly independent solutions \( y_1, \ldots, y_r \) of \( Ly = 0 \) satisfying the condition that the Wronskian \( W_r = W_r(y_1 \cdots y_r) \neq 0 \) on \( I \).

Proof. Suppose \( L = PQ \) where \( Q \) has order \( r \). Since any solution of \( Qy = 0 \) is also a solution of \( Ly = 0 \), we can choose any \( r \) linearly independent solutions of \( Qy = 0 \) and condition \( W_r \neq 0 \) will be satisfied.

On the other hand, assume \( Ly_i = 0 \) for \( i = 1, \ldots, r \) and \( W_r(y_1 \cdots y_r) \neq 0 \). Let

\[
Qy = \det \begin{bmatrix} y_1 & \cdots & y_r & y \\ y_1' & \cdots & y_r' & y' \\ \vdots & & \vdots & \vdots \\ y_1^{(r)} & \cdots & y_r^{(r)} & y^{(r)} \end{bmatrix} = q_0 y^{(r)} + q_1 y^{(r-1)} + \cdots + q_r y.
\]

Note that \( q_0 = W_r(y_1 \cdots y_r) \neq 0 \) and \( q_i \in C^{n+1-r-i} \). A direct computation shows that the coefficients \( p_i \) for \( i = 0, \ldots, n-r \) can be chosen successively so that the coefficients of \( y^{(n)}, y^{(n-1)}, \ldots, y^{(n-r)} \) in the product \( PQy \) where \( Py = p_0 y^{n-r} + \cdots + p_{n-r} y \) are \( s_0, s_1, \ldots, s_r \), respectively. Hence the operator \( N = L - PQ \) is of order less than \( r \). But \( y_1, \ldots, y_r \) are linearly independent solutions of \( Ny = 0 \). Hence \( N = 0 \) and \( L = PQ \).

Although the above theorem may have been known to Ince—see

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the discussion on pp. 119 and 126–127 in [1]—the author has not seen an explicit statement of it anywhere in the literature.

The conditions $W_r \neq 0$ for $r = 1, \ldots, n - 1$ are known to be necessary and sufficient for the factorability of an operator of type (*) into a "product" of first order operators—see [2].

References


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