ON THE STRUCTURE OF NONSTANDARD MODELS OF ARITHMETIC

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Abstract. In this paper we show that the additive group of each nonstandard model \( \ast \mathbb{Z} \) of the integers \( \mathbb{Z} \) is isomorphic to the group \( (F \times \mathbb{Z}, +) \) where \( F \) is a direct sum of \( \alpha \)-copies of the rationals \( \mathbb{Q} \), \( \alpha \) the cardinality of \( \ast \mathbb{Z} \), and \( + \) is defined by: \((a, x) + (b, y) = (a + b, x + y + g(a, b))\) for certain functions \( g \) mapping from \( F \times F \) to \( \mathbb{Z} \).

Introduction. Let \( \ast \mathbb{Z} \) denote any nonstandard model of arithmetic and set

\[ \ast \mathbb{Z} / \mathbb{Z} = \{ \bar{a} : a \in \ast \mathbb{Z} \}, \quad \bar{a} = a + \mathbb{Z}. \]

For each \( \ast \mathbb{Z} \) we let \( K = K(\ast \mathbb{Z}) \) denote the set of elements of \( \ast \mathbb{Z} \) of infinite height in \( \ast \mathbb{Z} \). (An element has infinite height if it is divisible by every positive integer.) It is easy to see that \( \bar{a} \cap K \) never contains more than one element. Kemeny [2] asked whether there existed a \( \ast \mathbb{Z} \) so that \( \bar{a} \cap K \) contained exactly one element for every \( a \in \ast \mathbb{Z} \). The importance of Kemeny's question lies in the fact that if such a \( \ast \mathbb{Z} \) existed, Goldbach's conjecture could be proven false; however, it was shown by Gandy [1] and Mendelson [3] that this question is answered in the negative.

The purpose of this paper is to analyze, in more detail, the structure of the models \( \ast \mathbb{Z} \), based on ideas suggested by Kemeny's and Mendelson's work and by the work of MacDowell and Specker [6]. The basic results about nonstandard models used in this paper can be found in Robinson [5]. We begin with Mendelson's solution to Kemeny's problem.

2. The additive group of \( \ast \mathbb{Z} \).

Theorem 1 (Mendelson). Any nonzero ring homomorphism of \( \ast \mathbb{Z} \) into itself is an order-preserving isomorphism.
Now suppose $K \cap a$ consists of exactly one element for each $a$ in some $*Z$. Then, clearly, the additive group of $*Z$ may be written as $K \oplus Z$. Hence, the map $f(x+n) = n$, where $x \in K$ and $n \in Z$, is not an order-preserving isomorphism, in contradiction to Theorem 1.

That the set $K$ plays an important role in determining the structure of $*Z$ is seen in the work of MacDowell and Specker. There, the additive group of $*Z$ is shown to be isomorphic to $K \oplus J$ where $J$ is some subgroup of the product group $\prod_{p=1}^{\infty} Z_p$, $Z_p$ being the integers mod $p$.

We now present a canonical form (Phillips [4]) for the additive group of each $*Z$ in which, essentially, only the addition varies from model to model. Given $*Z$, let $h:*Z/Z \to *Z$ be a choice function; that is, $h(\bar{a}) \in \bar{a}$. We "normalize" $h$ so that $h(0) = 0$. Then the map $\alpha_h$ given by

$$a \rightarrow (\bar{a}, a - h(\bar{a}))$$

is a one-to-one map of $*Z$ onto $*Z/Z \times Z$.

**Theorem 2.** Let $*Z/ZhZ$ denote the group whose elements are $*Z/Z \times Z$ and whose group operation is defined to be

$$(i) \quad (\bar{a}, x) + (\bar{b}, y) = (\bar{a} + \bar{b}, x + y + h(\bar{a}) + h(\bar{b}) - h(a + b)).$$

Then $\alpha_h$ is an additive isomorphism of $*Z$ onto $*Z/ZhZ$.

**Proof.** We have

$$a + b \rightarrow (\bar{a} + \bar{b}, a + b - h(a + b))$$

and hence we may define addition as

$$(\bar{a}, a - h(\bar{a})) + (\bar{b}, b - h(\bar{b})) = (\bar{a} + \bar{b}, a+b - h(a + b)).$$

It is easy to see that this addition is equivalent to that defined by (i) and the theorem follows.

**Definition 1.** Let $F$ be some direct sum of infinitely many copies of the rationals $Q$. By $F_0$ we mean the set of all functions $g:F \times F \to Z$ such that

1. $g(0, 0) = 0$,
2. $g(a, b) = g(b, a)$,
3. $g(a, b) + g(a+b, c) = g(a, b+c) + g(b, c)$.

If $g \in F_0$, we let $FgZ$ denote the set $F \times Z$ together with the additive operation defined as

$$(a, x) + (b, y) = (a + b, x + y + g(a, b)).$$
The next theorem is immediate.

**Theorem 3.** For each $g \in F_0$, $FgZ$ is an abelian group.

**Theorem 4.** For each $*Z$ of cardinal $\alpha$, $*Z/Z$ is isomorphic to a direct sum of $\alpha$ copies of the rationals $Q$, which in turn is isomorphic to $K(*Z)$.

**Proof.** See MacDowell-Specker [6].

**Theorem 5.** For each $*Z$ of cardinal $\alpha$ there exists a function $g \in F_0$, where $F$ is a direct sum of $\alpha$ copies of $Q$, so that the additive group of $*Z$ is isomorphic to $FgZ$; the isomorphism mapping $n \in Z$ onto $(0, n)$ in $FgZ$.

**Proof.** Let $\pi : F \rightarrow *Z/Z$ be the isomorphism given by Theorem 4 and let

$$g(a, b) = h(\pi(a)) + h(\pi(b)) - h(\pi(a) + \pi(b)),$$

where $h : *Z/Z \rightarrow *Z$ is a choice function, $h(\bar{0}) = 0$. Then $FgZ$ is isomorphic to $*Z/ZhZ$ by the map $(a, x) \rightarrow (\pi(a), x)$ and now the theorem follows from Theorem 2.

3. The set $F_1$. Let $F$ be an infinite direct sum of $\alpha$ copies of $Q$ and let $F_1 = F_1(\alpha)$ denote the set of all maps $g \in F_0$ such that for some nonstandard model $*Z$ of cardinal $\alpha$, the additive group of $*Z$ is isomorphic to $FgZ$ as in Theorem 5.

**Theorem 6.** $g \equiv 0$ is not in $F_1$.

**Proof.** Follows from Theorem 1. Hence, there are no linear choice functions from $*Z/Z \rightarrow *Z$.

Since $*Z/Z$ and $F$ are isomorphic, we now agree to identify corresponding elements in both structures.

**Theorem 7.** There exists a $g \in F_1$ such that $g(a, b) = 0$ for each $a, b \in F$ where $a \cap K \neq \emptyset$ and $b \cap K \neq \emptyset$. Hence $K \oplus Z$ is an additive subgroup of $*Z$.

**Proof.** If $x \in a \cap K$, set $h(a) = x$; define the choice function $h$ arbitrarily for the remaining $a \in F$. Then $h$ is linear on the set of $a$ where $a \cap K \neq \emptyset$.

Let $a = (a_1, a_2, \cdots)$ and $b = (b_1, b_2, \cdots)$. We set $e(a, b)$ equal to the finite cardinal of the set

$$E = (a, b) = \{i: \text{neither } a_i \text{ nor } b_i \text{ are integers or } a_i \neq -b_i\}.$$

**Theorem 8.** There exists a $g \in F_1$ such that $0 \leq |g(a, b)| \leq e(a, b)$ for all $a, b \in F$.
Proof. Let $A = \{x_1, \ldots, x_i, \ldots\}$ be a basis for $*Z/Z$ over $Q$ and we identify $r_1x_1 + \cdots + r_ix_i$ in $*Z/Z$ with $a \in F$ where $a_p = 0$ if $p < i$ or $p > j$ and $a_p = r_p$ if $i \leq p \leq j$. We define $h(x_i) \in x_i$ arbitrarily and if $a = \sum a_ix_i$ we set

$$h(a) = \sum_{p=i}^{j} [a_p h(x_p)]$$

where $[x]$ denotes the greatest integer function if $x \geq 0$ and $[x] = -[x]$ if $x < 0$ ($[x]$ can be defined by first embedding $*Z$ in $*Q$). $h$ can be shown to be a choice function on $*Z/Z \rightarrow *Z$ and hence we set

$$g(a, b) = h(a) + h(b) - h(a + b).$$

Thus

$$g(a, b) = \sum_{i \in E(a, b)} ([a_i h(x_i)] + [b_i h(x_i)] - [(a_i + b_i) h(x_i)]),$$

using familiar properties of $[x]$. The theorem now follows from the fact that for all $x$ and $y$,

$$[x] + [y] - [x + y] = \begin{cases} 1, \\ 0, \\ -1 \end{cases}$$

One might ask if instead of using choice functions from $F \rightarrow *Z$, could we define $g$ with simpler functions. The next theorem gives a partial answer to this question:

Theorem 9. If $h : F \rightarrow Z$ and $g(a, b) = h(a) + h(b) - h(a + b)$, then $g \in F_0 - F_1$.

Proof. It is not hard to show that in $FgZ$ we have

$$(0, n) \cdot (a/n, x_n) = (a, nx_n + nh(a/n) - h(a)).$$

Hence, for each $n$ let $x_n = -h(a/n)$. Then

$$(0, n) \cdot (a/n, x_n) = (a, -h(a))$$

for each $n$ and each $a$. Thus each element $(a, -h(a))$ has infinite height in $FgZ$ which implies $K \cap A \neq \emptyset$ for each $a \in *Z$ if the additive group of $*Z$ were isomorphic to $FgZ$.

Bibliography


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