CONTINUUM NEIGHBORHOODS AND FILTERBASES

DAVID P. BELLAMY1 AND HARVEY S. DAVIS2

Abstract. In this paper we prove that if \( \Gamma \) is a filterbase of closed subsets of a compact Hausdorff space then \( T(\cap \Gamma) = \bigcap \{ T(G) \mid G \in \Gamma \} \), where \( T(A) \) denotes the set of those points for which every neighborhood which is a continuum intersects \( A \) nonvoidly.

Introduction. In this paper \( S \) denotes a compact Hausdorff space. If \( p \in S \) and \( W \subset S \), then \( W \) is a continuum neighborhood of \( p \) iff \( W \) is a subcontinuum of \( S \) and \( p \in \text{Int}(W) \). If \( A \subset S \), \( T(A) \) denotes the complement of the set of those points \( p \) of \( A \) for which there exists a continuum neighborhood which is disjoint from \( A \) [1]. \( S \) is said to be \( T \)-additive iff for every collection \( \Lambda \) of closed subsets of \( S \) whose union is closed, \( T(\cup \Lambda) = \bigcup \{ T(L) \mid L \in \Lambda \} \) [2]. The following three theorems are established.

Theorem A. Let \( \Gamma \) be a filterbase of closed subsets of \( S \). Then \( T(\cap \Gamma) = \bigcap \{ T(G) \mid G \in \Gamma \} \).

Theorem B. \( S \) is \( T \)-additive iff for each pair \( A, B \) of closed subsets of \( S \), \( T(A \cup B) = T(A) \cup T(B) \).

Theorem C. Let \( A \) be a closed subset of \( S \). If \( K \) is a component of \( T(A) \) then \( T(A \cap K) = K \cup T(\emptyset) \).

Theorem A is used in establishing Theorems B and C. Theorem C is used to obtain the known result that if \( S \) and \( W \) are continua and \( W \subset S \) then \( W \) is a continuum [1].

Proof of Theorem A. It is immediate from the definition that whenever \( A \subset B \), \( T(A) \subset T(B) \) and thus \( T(\cap \Gamma) \subset \bigcap \{ T(G) \mid G \in \Gamma \} \).

Suppose \( p \in T(\cap \Gamma) \). There exists \( W \), a subcontinuum of \( S \), such that \( p \in \text{Int}(W) \) and \( W \cap (\cap \Gamma) = \emptyset \). Since \( W \) is compact, there exists a finite collection \( G_1, \ldots, G_n \) of elements of \( \Gamma \) whose intersection is disjoint from \( W \). By hypothesis there exists \( G \), an element of \( \Gamma \), which is contained in \( G_1 \cap \cdots \cap G_n \). Since \( G \) is disjoint from \( W \), \( p \in T(G) \). Hence \( p \in \bigcap \{ T(G) \mid G \in \Gamma \} \) and thus

Received by the editors January 6, 1970.

AMS 1969 subject classifications. Primary 5455; Secondary 5465.

Key words and phrases. Compact Hausdorff space, continuum neighborhood, \( T(A) \), \( T \)-additive, filterbase, component.

1 Research supported in part by the University of Delaware Research Foundation.
2 Research supported in part by the National Science Foundation NSF 71-1550.
\[ T(\cap \Gamma) = \cap \{ T(G) \mid G \in \Gamma \}. \]

**Proof of Theorem B.** The necessity of the condition is clear. Let \( \Lambda \) be a collection of closed subsets of \( S \) whose union is closed in \( S \). Since \( T(\cup \Lambda) \supseteq \bigcup \{ T(L) \mid L \in \Lambda \} \), it need only be shown that \( T(\cup \Lambda) \subseteq \bigcup \{ T(L) \mid L \in \Lambda \} \) in order to establish the sufficiency of the condition.

Suppose \( x \in \bigcup \{ T(L) \mid L \in \Lambda \} \). Then for each \( L \in \Lambda \) let \( F(L) \) be the collection of closed subsets \( A \) of \( S \) such that \( L \subseteq \text{Int}(A) \). If \( L = \emptyset \), clearly \( T(L) = \bigcap \{ T(A) \mid A \in F(L) \} \). If \( L \neq \emptyset \), then \( F(L) \) is a filterbase of closed subsets of \( S \) and, since \( \cap F(L) = L \), \( T(L) = \bigcap \{ T(A) \mid A \in F(L) \} \) by Theorem A.

Hence, for each \( L \), \( x \in \bigcap \{ T(A) \mid A \in F(L) \} \) and thus there exists, for each \( L \), \( x \in F(L) \), such that \( x \in T(f(L)) \). \( \{ \text{Int}(f(L)) \mid L \in \Lambda \} \) is an open covering of \( \cup \Lambda \). Since \( \cup \Lambda \) is compact there exists a finite subcollection \( \Gamma \) of \( \{ f(L) \mid L \in \Lambda \} \) such that \( \cup \{ \text{Int}(f(L)) \mid L \in \Gamma \} \subseteq \cup \{ T(G) \mid G \in \Gamma \} \). Since for all \( G \in \Gamma \), \( x \in T(G) \), it follows that \( x \in T(\cup \Lambda) \). Thus \( T(\cup \Lambda) \subseteq \bigcup \{ T(L) \mid L \in \Lambda \} \).

**Proof of Theorem C.** Two technical lemmas are established. Theorem C follows easily from these two lemmas and Theorem A.

**Lemma 1.** Let \( A \) be a subset of \( S \). \( p \in S - T(A) \) iff there is a subcontinuum \( W \) and an open subset \( Q \) of \( S \) such that \( p \in \text{Int}(W) \cap Q \), \( \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap A \cap Q = \emptyset \).

**Proof.** Let \( p \in S - T(A) \). There is a subcontinuum \( W \) of \( S \) such that \( p \in \text{Int}(W) \) and \( W \cap A = \emptyset \). Since \( S \) is regular there is an open subset \( Q \) of \( S \) such that \( p \in Q \) and \( \text{Cl}(Q) \subseteq \text{Int}(W) \). It is clear that \( \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap A \cap Q = \emptyset \).

Now suppose that there is a subcontinuum \( W \) and an open subset \( Q \) of \( S \) such that \( p \in \text{Int}(W) \cap Q \), \( \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap A \cap Q = \emptyset \). Since \( \text{Fr}(Q) \) is compact and disjoint from \( T(A) \), there exists a finite collection \( \{ W_i \} \) of subcontinua of \( S \), all disjoint from \( A \), such that \( \cup \{ \text{Int}(W_i) \} \supset \text{Fr}(Q) \). Since if \( W \subseteq Q \) it is immediate that \( p \in S - T(A) \), assume \( W \cap S - Q \neq \emptyset \). The closure of each component of \( W \cap Q \) must intersect at least one of the \( W_i \)'s, since \( \text{Fr}(Q) \subseteq \cup \{ W_i \} \).

Hence \( (W \cap Q) \cup (\cup \{ W_i \}) = H \) has only a finite number of components. Since \( p \in \text{Int}(W) \cap Q \), there is a component \( K \) of \( H \) such that \( p \in \text{Int}(K) \) and, of course, \( K \cap A \subseteq H \cap A = \emptyset \). Thus \( p \in S - T(A) \).

**Lemma 2.** Let \( A \) be a subset of \( S \). If \( T(A) = M \cup N \) separate then \( T(A \cap M) = M \cup T(\emptyset) \).
Proof. Suppose $p \in T(A \cap M) - (M \cup T(\varnothing))$. Since $p \in T(\varnothing)$, there is a subcontinuum $W$ of $S$ such that $p \in \text{Int}(W)$. Since $S$ is normal, there is an open subset $Q$ of $S$ containing $N$ whose closure is disjoint from $M$. It is clear that $p \in \text{Int}(W) \cap Q$, $\text{Fr}(Q) \cap T(A \cap M) \subseteq \text{Fr}(Q) \cap T(A) = \varnothing$ and $W \cap (A \cap M) \cap Q \cap Q \cap M = \varnothing$. Hence, by Lemma 1, $p \in T(A \cap M)$, thus contradicting the supposition.

Now suppose that $p \in (M \cup T(\varnothing)) - T(A \cap M)$. Since $p \in T(A \cap M)$ and $\varnothing \subseteq A \cap M$, $p \in T(\varnothing)$. Hence $p \in M$. There is an open subset $Q$ of $S$ containing $M$ whose closure is disjoint from $N$. Since $p \in T(A \cap M)$, there is a subcontinuum $W$ of $S$ such that $p \in \text{Int}(W) \cap Q$ and $W \cap (A \cap M) = \varnothing$. It is clear that $p \in \text{Int}(W) \cap Q$ and $\text{Fr}(Q) \cap T(A) = \varnothing$. Since $Q \cap N = \varnothing$, $W \cap A \cap Q = W \cap (A \cap M) = \varnothing$. Hence, by Lemma 1, $p \in T(A)$ so $p \in M$, thus contradicting the supposition.

Now in order to establish Theorem C, let $A$ be a closed subset of $S$ and $K$ be a component of $T(A)$. Let $\{K_a\}$ be the collection of all subsets of $T(A)$ such that $K \in \{K_a\}$ and $K_a$ is both open and closed in $T(A)$. Note that the collection $\{A \cap K_a\}$ can only fail to be a filter-base if for some $K_a$, $A \cap K_a = \varnothing$. In this case the conclusion of Theorem A is trivial. Lemma 2, of course, remains true even if $A \cap M = \varnothing$ so, for each $K_a$, $T(A \cap K_a) = K_a \cup T(\varnothing)$. That this can occur is seen by letting $S$ be the Cantor set, $A$ be the void set and $K_a$ be $S$.

The following sequence of equalities establish the theorem:

\[
T(A \cap K) = T(\cap \{A \cap K_a\}) = \cap \{T(A \cap K_a)\} = \cap \{K_a \cup T(\varnothing)\} = \cap \{K_a\} \cup T(\varnothing) = K \cup T(\varnothing).
\]

Theorem C is not true if the requirement that $A$ be closed is dropped. Let $S$ be the unit interval and let $A$ be the sequence $\{1/n\}$. Then $T(A) = \{0\} \cup A$. Let $K = \{0\}$. Then $T(A \cap K) = T(\varnothing)$ which is void since $S$ is a continuum. But $K \cup T(\varnothing)$ is not void.

Corollary 1. Let $S$ be a continuum and $W$ be a subcontinuum of $S$. $T(W)$ is a subcontinuum of $S$.

Proof. Suppose $T(W) = A \cup B$ separate. By Theorem C, $T(W \cap A) = A$ and $T(W \cap B) = B$ since $T(\varnothing) = \varnothing$ when $S$ is a continuum. $W \cap A \neq \varnothing$ since $T(W \cap A) \neq \varnothing$ and, likewise $W \cap B \neq \varnothing$. Hence $W = (W \cap A) \cup (W \cap B)$ separate, contradicting the hypothesis and thus establishing the proposition.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Corollary 2. Let $S$ be a continuum and let $W_1$ and $W_2$ be subcontinua of $S$. If $T(W_1 \cup W_2) \neq T(W_1) \cup T(W_2)$ then $T(W_1 \cup W_2)$ is a continuum.

Proof. Suppose $T(W_1 \cup W_2) = A \cup B$ separate. By Lemma 2, $T((W_1 \cup W_2) \cap A) = A$ and $T((W_1 \cup W_2) \cap B) = B$. Suppose $W_1 \subset A$. If $W_2 \subset A$ then $A = T((W_1 \cup W_2) \cap A) = T(W_1 \cup W_2)$, thus contradicting the supposition. Hence $W_2 \subset B$. But then $T(W_1) = A$ and $T(W_2) = B$. Thus $T(W_1 \cup W_2) = T(W_1) \cup T(W_2)$. Corollaries 1 and 2 are special cases of Theorem 8 of [1].

Corollary 3. Let $S$ be a continuum and let $A$ and $B$ be closed subsets of $S$. If $K$ is a component of $T(A \cup B)$ which lies in neither $T(A)$ nor $T(B)$, then, $K \cap A \neq \emptyset \neq K \cap B$.

Proof. Since $S$ is a continuum, $T(\emptyset) = \emptyset$ and, by Theorem C, $T((A \cup B) \cap K) = K$. Since $K$ lies in neither $T(A)$ nor $T(B)$, $(A \cup B) \cap K$ meets both $A$ and $B$. Thus $K$ meets both $A$ and $B$.

Corollary 4. Let $S$ be a continuum and let $A$ and $B$ be closed subsets of $S$. If $T(A \cup B) \neq T(A) \cup T(B)$ then there exists a subcontinuum $K \subset T(A \cup B)$ such that $K \cap A \neq \emptyset \neq K \cap B$.

Proof. Let $K$ be the component of some point in $T(A \cup B) - (T(A) \cup T(B))$ and apply Corollary 3.

Bibliography


Michigan State University, East Lansing, Michigan 48823

University of Delaware, Newark, Delaware 19711