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QUASI-JORDANIAN CONTINUA[1]

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ABSTRACT. A generalization of ordinary closed Jordan curves is obtained by considering nondegenerate compact continua which form the boundary of a simply connected open subset D of Moore space such that all crosscuts of D disconnect its closure. Separation in such continua by points of accessibility and the relation of such continua to their (possibly infinitely many) complementary domains are studied.

INTRODUCTION. Suppose that M is a nondegenerate compact continuum which bounds a simply connected domain D, all of whose crosscuts disconnect $\overline{D}$. We show that such continua have a connectivity structure much like that of closed Jordan curves. However, even in the Euclidean plane, such continua may possess infinitely many complementary domains.

DEFINITIONS AND NOTATION. S will denote a space satisfying R. L. Moore's Axioms 0, 1, 2, 3, 4, and 5 [1]. If X and Y are point sets: $X \cup Y$ denotes their union; $X \cap Y$ denotes their common part; $\overline{X}$ is the closure of X (in S); and $\mathrm{Bd}(X)$ denotes the boundary of X. A *complementary domain* of a closed subset K of S is a component of $S-K$. *A crosscut* [*endcut*] of a connected open subset D of S is an arc both of whose ends [only one of whose ends] lie on the boundary of D, and whose remaining points are points of D. Any point of $\mathrm{Bd}(D)$ that is an endpoint of an endcut of D is said to be *accessible from* D. A domain D is *simply connected* if and only if it is connected and contains a complementary domain of every simple closed curve lying in it. A compact nondegenerate continuum M will be called *quasi-Jordanian* if and only if there exists a simply connected domain D such that (1) $\mathrm{Bd}(D)=M$ and (2) every crosscut of D disconnects $\overline{D}$.

CONVENTIONS. For simplicity, the letter M will consistently be employed to denote a quasi-Jordanian continuum lying in S and D will denote a complementary domain of M satisfying (1) and (2).

THEOREM 1. *If* 0 *is a point of* M *and* J *is a simple closed curve lying in* $D\cup 0$, *then* D *contains a complementary domain of* J.

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Proof. The conclusion is automatic if J does not pass through 0. Suppose, however, that 0 is a point of J and that D contains neither complementary domain of J. Then J separates two points of M from each other. Letting I and E denote the two complementary domains of J, it may be shown that there exist points $A \in I \cap \mathrm{Bd}(D)$ and $B \in E \cap \mathrm{Bd}(D)$ and a crosscut AXB of D such that $J \cap AXB = X$. Now any point belonging to $(I \cap D) - AX$ may be joined to $J - X$ by an arc lying in D and missing AXB, since $D - AX$ is connected [**1**, p. 173, Theorem 20]. A similar process may be carried out for points of $(E \cap D) - XB$. It follows that $[D - AXB] \cup (J - X)$ is connected. But the latter set is dense in $\overline{D} - AXB$, contradicting the fact that AXB disconnects $\overline{D}$. Hence D contains either I or E.

Theorem 2. *There exists only one complementary domain E of M, distinct from D, whose boundary is the whole of M.*

Proof. Suppose M has a cutpoint 0. Then there exists a simple closed curve J such that (1) $M \cap J = 0$ and (2) J separates two points of M from each other [**1**, p. 202, Theorem 53]. But then J lies in $D \cap 0$, contradicting Theorem 1. Hence M has no cutpoint. Now let AB be a crosscut of D. Since AB disconnects $\overline{D}$, it may be seen that $\overline{D} - AB$ is the sum of two mutually separated connected sets P and Q such that $\overline{P} \cap \overline{Q} = AB$. With the aid of the latter equation and the observation that AB does not separate S [**1**, p. 175, Theorem 21], it is clear that $A \cup B$ separates two points C and F from each other in M. Furthermore, since M has no cutpoint, $A \cup B$ is irreducible with respect to the property of separating C from F in M. Hence there exists a simple closed curve J such that (1) J separates C from F and (2) $J \cap M = A \cup B$ [**1**, p. 202, Theorem 53]. Denote by AXB and AYB the two arcs on J with endpoints as indicated. An application of Theorem 1 shows that D does not contain $J - (A \cup B)$ and since J separates two points of M from each other, it is clear that $S - \overline{D}$ does not contain $J - (A \cup B)$. Hence we may suppose that one of the two arcs AXB and AYB, say AXB, lies, with the exception of its ends, wholly in D, and the other, say AYB, lies, with the exception of its ends, wholly in $S - \overline{D}$. Let E denote the complementary domain of M containing Y. Note that A and B are accessible points of M lying on $\mathrm{Bd}(E)$. Now let A_1 and B_1 denote any two points of M which are accessible from D and lie in distinct complementary domains of J. Then there exists an arc $A_1Y_1B_1$ such that $S - \overline{D}$ contains all of $A_1Y_1B_1$ except its ends. Clearly, $A_1Y_1B_1$ and AYB intersect. Hence A_1 and B_1 lie on $\mathrm{Bd}(E)$. It follows that each point of M which is accessible from D is a point of $\mathrm{Bd}(E)$. Hence $\mathrm{Bd}(E) = M$. Clearly, E is the only component of $S-\overline{D}$ with this property. For if E' is any other, then the connected set $C\cup E'\cup F$ intersects both complementary domains of J. Hence E' intersects E.

EXAMPLE. On p. 119 of [2], there is an illustration of a connected planar region which spirals around a circular disk. The boundary M of this region is quasi-Jordanian and has three (3) complementary domains. Other examples are easily obtainable of quasi-Jordanian continua with infinitely many complementary domains.

THEOREM 3. *No proper subcontinuum of M disconnects M. In particular, M has no cutpoint.*

PROOF. This is immediate since by Theorem 2, M is the outer boundary of D relative to E [1, p. 178, Theorem 128].

THEOREM 4. *If A and B are points of M which are accessible from D, then $M-(A\cup B)$ is the sum of two mutually separated connected sets H and K such that $\overline{H}=H\cup A\cup B$ and $\overline{K}=K\cup A\cup B$.*

PROOF. Consider a crosscut AB of D. Then $\overline{D}-AB$ is the sum of two mutually separated connected sets P and Q such that $\overline{P}\cap\overline{Q}=AB$. Let $H=\mathrm{Bd}(P)-AB$ and $K=\mathrm{Bd}(Q)-AB$. Clearly, H and K are mutually separated subsets of M and $M-(A\cup B)=H\cup K$. Theorem 3 implies that H and K are connected. Finally, since M has no cutpoint, $\overline{H}=H\cup A\cup B$ and $\overline{K}=K\cup A\cup B$.

THEOREM 5. *The continua $H\cup A\cup B$ and $K\cup A\cup B$ are irreducible from A to B.*

PROOF. This is an immediate consequence of Theorems 3 and 4.

THEOREM 6. *If C is a point of the set H which is accessible from D, then $H-C$ is the sum of two mutually separated connected sets U and V such that $\overline{U}=U\cup A\cup C$ and $\overline{V}=V\cup B\cup C$.*

PROOF. By Theorem 4, $M-(A\cup C)$ is the sum of two mutually separated connected sets U and W_1 such that $\overline{U}=U\cup A\cup C$ and $\overline{W}_1=W_1\cup B\cup C$. Suppose for convenience that W_1 contains the connected set $K\cup B$. Then U is a connected subset of $H-C$. Also, $M-(B\cup C)$ is the sum of two mutually separated connected sets V and W_2 such that $\overline{V}=V\cup B\cup C$ and $\overline{W}_2=W_2\cup B\cup C$, and we suppose that W_2 contains the connected set $K\cup A\cup U$. Hence U and V are two mutually separated connected subsets of $H-C$ and $A\cup U\cup C\cup V\cup B$ is a continuum. By Theorem 5, $H\cup A\cup B=A\cup U\cup C\cup V\cup B$. Hence $H-C=U\cup V$.

THEOREM 7. *If A, B, C, and F are four points of M which are accessible from D and $A \cup B$ separates C from F in M, then $C \cup F$ separates A from B in M.*

PROOF. This is an immediate consequence of Theorems 4 and 6.

THEOREM 8. *If every point of M is accessible from D, then M is a simple closed curve.*

PROOF. By Theorem 4, each pair of points of M disconnects M. Hence M is a simple closed curve.

THEOREM 9. *If M_1 is the boundary of a complementary domain of M distinct from both D and E, then M_1 is a continuum of condensation of M.*

PROOF. Suppose, to the contrary, that $(\overline{M - M_1})$ does not contain M_1. Then there exist two points A and B of M_1 which are accessible from D. Let H and K denote the two components of $M-(A \cup B)$. By the proof of Theorem 2, either $\overline{H}$ contains M_1 or $\overline{K}$ contains M_1. Assume that $\overline{H} = H \cup A \cup B$ contains M_1. As a consequence of our original assumption, there exist two points C and F belonging to $H \cap M_1$ which are accessible from D. By Theorems 4 and 6, $H - (C \cup F)$ is the sum of three mutually separated connected sets, U, W, and V, such that $\overline{U} = U \cup A \cup C$, $\overline{W} = W \cup C \cup F$, and $\overline{V} = V \cup F \cup B$. But since M_1 does not intersect both components of $M-(C \cup F)$, it follows that M_1 is a subset of $\overline{U} \cup \overline{V}$. This is a contradiction, for the latter sets are mutually separated.

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