ONE-SIDED BOUNDARY BEHAVIOR FOR CERTAIN HARMONIC FUNCTIONS

T. K. BOEHME AND MAX L. WEISS

Abstract. Some results concerning the maximal ideal space of \( H^\infty \) of the disk are applied to harmonic functions. The methods yield a Lindelöf type theorem for harmonic functions and extend to bounded harmonic functions a criterion of Tanaka which is necessary and sufficient in order that the boundary value function be one-sided approximately continuous.

1. Introduction. We are concerned in this paper with connections between the one-sided behavior of an \( L^\infty \) function at a point of the unit circle, \( C = \{z: |z| = 1\} \), and the boundary behavior of the harmonic extension of the function into the unit disc, \( D = \{z: |z| < 1\} \). Our techniques consist mainly of combining certain concrete estimates for harmonic measures with some facts about the Banach algebra, \( H^\infty \), of bounded analytic functions on \( D \). We assume for this latter area that the reader is familiar with the contents of Chapter 10 of Hoffman's book, [3], and with [4] and [5].

The main focal point is the introduction of a class of homomorphisms in the maximal ideal space of \( H^\infty \) which we call the "barely tangential homomorphisms." These homomorphisms play a role for the one-sided boundary behavior of \( L^\infty \) functions similar to that played by the radial homomorphisms for two-sided behavior in [1].

§2 is mainly devoted to an intrinsic study of the barely tangential homomorphisms. In §3 we obtain a theorem (Theorem 3.1) characterizing one-sided approximate continuity from above of an \( L^\infty \) function in terms of the behavior of the function on the supports of the representing measures of upper barely tangential homomorphisms. Subsequently, we easily obtain a result of Tanaka [6, Theorem 5] characterizing one-sided approximate continuity. As a final result we prove a "Lindelöf-type theorem" for \( L^\infty \) functions.

2. Barely tangential homomorphisms. We first recall that the collection, \( H^\infty \), of bounded analytic functions on \( D \) forms a function algebra with pointwise operations and the supremum norm. Its maximal ideal space, \( \mathbb{D} \), is a compactification of \( D \), [3]. Every homomorphism in \( \mathbb{D} \) can be approached by a universal net in \( D \) or by one...
in any dense subset of $D$. One can also represent $H^\infty$ as a subalgebra of the Banach algebra, $L^\infty$, of all bounded measurable functions on $C$. We rely on the references for a complete description of these connections. We remark here that for any $L^\infty$ function $f$ on $C$ we will continue to denote as "$f$" any of the standard representations of $f$ on $D$, $\mathcal{D}$, or $C$.

For simplicity we restrict our attention to the fiber, $\mathcal{D}_1$, above 1, i.e., those homomorphisms which are approached by nets tending to 1. From here on out we will simply assume phrases such as, "at 1." The collection, $\mathcal{S}$, of such homomorphisms which are approached within a Stolz angle are called the Stolz homomorphisms. In [5] the $w^*$-closure, $\mathcal{S}^* = \mathcal{L}$, of the Stolz homomorphisms is called the Lindelöf homomorphisms.

**Definition 2.1.** Let $\mathcal{S}$ and $\mathcal{L}$ be the Stolz and Lindelöf homomorphisms, respectively. Then, the collection, $\mathcal{B} = \mathcal{L} - \mathcal{S}$, is called the barely tangential homomorphisms. Those points, $\mathcal{B}^+$ [$\mathcal{B}^-$], of $\mathcal{B}$ which are approached by nets tending to 1 tangentially from above [below] are termed the upper [lower] barely tangential homomorphisms.

Our first result shows a relationship between $\mathcal{B}$ and $\mathcal{L}$ which we shall use once in §3. We recall that for any function algebra, $A$, on a compact Hausdorff space $X$, each point $h \in X$ has a representing measure, $\mu_h$, spread on the Silov boundary of $A$.

**Lemma 2.2.** Let $A$ denote the restriction algebra of $H^\infty$ to the Lindelöf homomorphisms, $\mathcal{L}$. Then, the Silov boundary of $A$ is contained in the barely tangential homomorphisms, $\mathcal{B}$. If $h$ is a Stolz homomorphism, then for any representing measure $\mu_h$ we have $\mu_h(\mathcal{B}^+) > 0$ and $\mu_h(\mathcal{B}^-) > 0$. In particular, if $f \in H^\infty$, then $f$ has radial limit $\alpha$ if and only if $f$ is constantly $\alpha$ on $\mathcal{B}^+$ (or on $\mathcal{B}^-$).

**Proof.** We show that no Stolz homomorphism is contained in the Silov boundary. Let $h \in \mathcal{S}$. Since by [5, §3] $\mathcal{S}$ is open in $\mathcal{D}_1$ and since $\mathcal{B}$ is compact there is a neighborhood $N$ of $h$ such that $N \subseteq \mathcal{S}$ and $N \cap \mathcal{B} = \emptyset$. Now, suppose $f \in H^\infty$ and $f$ peaks on $N$, i.e., $|f(h_1)| = \|f\|_\mathcal{S}$ for some $h_1 \in N$. Choose some Gleason part, $P$, of $\mathcal{D}$ which contains such a point in $N$. Using the results of [5, especially §6] we see that $P \subseteq \mathcal{S}$ and $P \cap \mathcal{B} \neq \emptyset$. Since the restriction of $f$ to $P$ is analytic and achieves its maximum modulus on $P$ it is constant. Thus $f$ peaks at some point of $P \cap \mathcal{B}$. Consequently, we have shown that $f$ peaks outside $N$. Therefore, $h$ cannot be in the Silov boundary and the latter is contained in $\mathcal{B}$.

Next, let $h \in \mathcal{S}$ and let $\mu_h$ be a representing measure. Let $\nu$ be the harmonic measure of $C^+ = \{e^{i\theta} : 0 \leq \theta \leq \pi\}$. Let $\nu$ be a harmonic con-
jugate and let \( f = \exp(u - iv) \). Then \(|f| = e\) on \( \Omega^+ \), \(|f| = 1\) on \( \Omega^-\) and \(1 < |f| < e\) on \( \mathcal{S} \). If \( \mu_h(\Omega^+) = 0\), then we would have \(|f(h)| \leq 1\) which is impossible. Thus \( \mu_h(\Omega^+) > 0\), and similarly \( \mu_h(\Omega^-) > 0\).

Finally, suppose \( f \in H^\infty\). If \( f \) has radial limit \( \alpha \), then \( f \) has Stolz limit \( \alpha \) and thus \( f = \alpha \) on \( \mathcal{S} \). Consequently, by continuity \( f = \alpha \) on \( \Omega^+ \) (or \( \Omega^-\)). Conversely, suppose \( f = \alpha \) on \( \Omega^+ \) (or \( \Omega^-\)). Let \( \mu_h \) be a Jensen measure for \( h \in \mathcal{S} \). Then,

\[
\log |f(h)| - \alpha | \leq \int_{\Omega^+ \cup \Omega^-} \log |f - \alpha| \, d\mu_h.
\]

Since \( \mu_h(\Omega^+) > 0\) and \( f - \alpha = 0 \) on \( \Omega^+\), the integral above equals \( -\infty \) and so \( f(h) = \alpha \). Thus, \( f = \alpha \) on \( \mathcal{S} \) and \( a \) fortiori \( f \) tends to \( \alpha \) radially.

In order to avoid exactly similar cases we now concentrate entirely on the upper barely tangential homomorphisms.

We next define a cluster set which will allow us to state more concretely Theorem 3.3 in \$3\$. Each Stolz angle approach can be written as \( \theta = c(1 - r) \) and as \( c \) increases the approach tends more and more toward an upper tangential approach. Given \( d \leq c \) we denote by \( S_{d,c} \) the collection of all homomorphisms in \( \mathcal{D}_1 \) which are approached by nets corresponding to all Stolz angle approaches, \( a \), such that \( d \leq a \leq c \). Given \( f \in L^\infty \) define \( S_{d,c}(f, 1) \) as the collection of all cluster values of \( f(re^{i\theta}) \) as \( re^{i\theta} \rightarrow 1 \) and \( d \leq \lim \inf \theta/(1 - r) \leq \lim \sup \theta/(1 - r) \leq c \). It is not hard to see that we then have \( S_{d,c}(f, 1) = f[S_{d,c}] \). Now, define \( B^+(f, 1) \), the upper barely tangential cluster set of \( f \) at \( 1 \) by

\[
B^+(f, 1) = \bigcap_{d > 0} \left[ \bigcup_{c > 0} S_{d,c}(f, 1) \right].
\]

**Lemma 2.3.** Let \( f \in L^\infty \). Then, \( B^+(f, 1) = f[\Omega^+] \).

**Proof.** Suppose \( h_0 \in \Omega^+ \). Then, there is a net \( h_\alpha \in \mathcal{S} \) such that \( h_\alpha \rightarrow h_0 \). Clearly, for each \( d \geq 0 \), \( h_\alpha \) is eventually in \( \bigcup_{c \geq 0} S_{d,c} \). Thus, \( h_0 \in \bigcap_{d \geq 0} \left[ \bigcup_{c \geq 0} S_{d,c} \right]^{-} \). On the other hand suppose \( h_0 \in \bigcap_{d \geq 0} \left[ \bigcup_{c \geq 0} S_{d,c} \right]^{-} \). Then, for each \( d \geq 0 \), \( h_0 \in \bigcup_{c \geq 0} S_{d,c} \). Since \( \bigcup_{c \geq 0} S_{d,c}^{-} = \Omega^+ \) for each \( d \), \( h_0 \in \Omega^+ \). Therefore,

\[
\Omega^+ = \bigcap_{d \geq 0} \left[ \bigcup_{c \geq 0} S_{d,c} \right]^{-}.
\]

Since the intersection is over a nested system of compact sets and since \( f \) is continuous on \( \mathcal{D} \) we have

\[
f[\Omega^+] = \bigcap_{d \geq 0} \left[ \bigcup_{c \geq 0} f[S_{d,c}] \right]^{-} = B^+(f, 1).
\]
Given a measurable subset, $M$, of the circle, $C$, we let $u_M$ denote the harmonic measure of $C$,

$$u_M(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_M(e^{i\phi}) \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\phi,$$

where $\chi_M$ is the characteristic function of $M$. If $M$ is a subset of the upper portion of the unit circle, $C^+ = \{ e^{i\theta} : 0 \leq \theta \leq \pi \}$, we let

$$d(M) = \lim \inf_{\theta \to 0^+} \frac{1}{\theta} \int_0^\theta \chi_M(e^{i\phi}) d\phi, \quad D(M) = \lim \sup_{\theta \to 0^+} \frac{1}{\theta} \int_0^\theta \chi_M(e^{i\phi}) d\phi$$

denote the lower and upper densities of $M$ at 1 from above.

Our basic results rely heavily on the following estimates for harmonic measures.

**Lemma 2.4.** Let $M$ be a measurable subset of $C^+$. If $d(M) = D(M)$, then for every $h \in \mathbb{S}^+$, $u_M(h) = d(M)$.

**Lemma 2.5.** Let $M$ be a measurable subset of $C^+$. Then, there exists an $h \in \mathbb{S}^+$ with

$$u_M(h) \geq \frac{2}{\pi} \tan^{-1} \frac{D(M)}{2\sqrt{1 - D(M)}}.$$

**Proof of 2.4.** Let $\epsilon > 0$ and choose $0 < \theta_0 < \pi/2$ so that, for $0 \leq \theta \leq \theta_0$, $d(M) - \epsilon < \frac{1}{\theta} \int_0^\theta \chi_M(e^{i\phi}) d\phi \leq d(M) + \epsilon$.

Since the harmonic measure of $M \cap [\theta_0, \pi]$ tends to zero as $\varepsilon \to 1$ we have, as $\varepsilon \to 1$,

$$u_M(re^{i\theta}) = o(1) + \frac{1}{2\pi} \int_0^{\theta_0} P_r(\theta - \phi) \chi_M(e^{i\phi}) d\phi,$$

where $P_r(t)$ is the Poisson kernel. Integrating by parts we have

$$\frac{1}{2\pi} \int_0^{\theta_0} P_r(\theta - t) \chi_M(e^{i\phi}) dt = \frac{P_r(\theta - \theta_0)}{2\pi} \int_0^{\theta_0} \chi_M(e^{i\phi}) ds + \frac{1}{2\pi} \int_0^{\theta_0} P'_r(\theta - t) \int_0^t \chi_M(e^{i\phi}) ds dt.$$

Thus, as $\varepsilon \to 1$,

$$u_M(re^{i\theta}) = o(1) + \frac{1}{2\pi} \int_0^{\theta_0} P'_r(\theta - t) \int_0^t \chi_M(e^{i\phi}) ds dt.$$
For \(0 \leq t \leq \theta < \pi/2\), \(P_t^\prime (\theta - t) \leq 0\) so
\[
(d(M) + \epsilon)tP_t^\prime (\theta - t) \leq P_t^\prime (\theta - t) \int_0^t \chi_M(e^{is})ds \leq (d(M) - \epsilon)tP_t^\prime (\theta - t);
\]
while, for \(\theta \leq t \leq \theta_0\), \(P_t^\prime (\theta - t) \geq 0\) so
\[
(d(M) - \epsilon)tP_t^\prime (\theta - t) \leq P_t^\prime (\theta - t) \int_0^t \chi_M(e^{is})ds \leq (d(M) + \epsilon)tP_t^\prime (\theta - t).
\]
Therefore,
\[
o(1) + \frac{d(M) + \epsilon}{2\pi} \int_0^\theta tP_t^\prime (\theta - t)dt + \frac{d(M) - \epsilon}{2\pi} \int_\theta^{\theta_0} tP_t^\prime (\theta - t)dt
\]
\[
\leq u_M(re^{i\theta})
\]
\[
\leq o(1) + \frac{d(M) - \epsilon}{2\pi} \int_0^\theta tP_t^\prime (\theta - t)dt + \frac{d(M) + \epsilon}{2\pi} \int_\theta^{\theta_0} tP_t^\prime (\theta - t)dt.
\]
Now,
\[
\int_0^\theta tP_t^\prime (\theta - t)dt = \frac{-(1+r)\theta}{1-r} + 2 \tan^{-1}\left\{\frac{1+r}{1-r} \tan \frac{\theta}{2}\right\},
\]
\[
\int_\theta^{\theta_0} tP_t^\prime (\theta - t)dt = o(1) + \frac{(1+r)\theta}{1-r} + 2 \tan^{-1}\left\{\frac{1+r}{1-r} \tan \frac{\theta_0 - \theta}{2}\right\},
\]
so that if \(re^{i\theta} \to 1\) in such a way that \(\theta/(1-r) \to c \geq 0\), we have
\[
(d(M) + \epsilon) \left[\frac{-c}{\pi} + \frac{\tan^{-1} c}{\pi}\right] + (d(M) - \epsilon) \left[\frac{c}{\pi} + \frac{1}{2}\right]
\]
\[
\leq \lim \inf u_M(re^{i\theta}) \leq \lim \sup u_M(re^{i\theta})
\]
\[
\leq (d(M) - \epsilon) \left[\frac{-c}{\pi} + \frac{\tan^{-1} c}{\pi}\right] + (d(M) + \epsilon) \left[\frac{c}{\pi} + \frac{1}{2}\right] .
\]
Since this is true for every \(\epsilon > 0\), we have as \(re^{i\theta} \to 1\), \(\theta/(1-r) \to c \geq 0\),
\[
d(M) \left[\frac{1}{2} + \frac{\tan^{-1} c}{\pi}\right]
\]
\[
\leq \lim \inf u_M(re^{i\theta}) \leq \lim \sup u_M(re^{i\theta}) \leq d(M) \left[\frac{1}{2} + \frac{\tan^{-1} c}{\pi}\right] .
\]
Thus, as \(re^{i\theta} \to 1\), \(\theta/(1-r) \to c \geq 0\),
Thus, for any homomorphism, $h_c$, in $\mathcal{S}_{c,e}$ we have

$$u_M(h_c) = d(M) \left[ \frac{1}{2} + \frac{\tan^{-1} c}{\pi} \right].$$

If $h \in \mathfrak{g}^+$, it is the limit of homomorphisms $h_c$ for which $c \to \infty$. Thus, because $u_M$ is continuous we have, letting $c \to \infty$, $u_M(h) = d(M)$, as claimed.

**Proof of 2.5.** Let $D = D(M)$, let $\epsilon > 0$, and pick $\theta_n \to 0$ such that

$$\frac{1}{2\theta_n} \int_0^{2\pi} \chi_M(e^{it}) dt \geq D - \epsilon.$$

Then,

$$u_M(r e^{i\theta_n}) \geq \int_0^{2\pi} P_r(\theta_n - t) \chi_M(e^{it}) dt$$

$$\geq 2 \int_0^{(D-\epsilon)\theta_n} P_r(\theta_n - t) dt = 2 \int_{(1-D+\epsilon)\theta_n}^{\theta_n} P_r(t) dt$$

$$= \frac{2}{\pi} \tan^{-1} \left( \frac{1 + r}{1 - r} \left( \frac{\theta_n}{2} - \tan \frac{(1 - D + \epsilon)\theta_n}{2} \right) \right).$$

Let $\tau_n$ be determined by $\theta_n = c(1 - \tau_n)$. Then, the limit as $\theta_n \to 0$ of the numerator of the argument of $\tan^{-1}$ in the last expression is $c(D - \epsilon)$, while that of the denominator is $1 + c^2(1 - D + \epsilon)$. Hence,

$$\limsup_{\theta_n \to 0} u_M(r e^{i\theta_n}) \geq \frac{2}{\pi} \tan^{-1} \left\{ \frac{c(D - \epsilon)}{1 + c^2(1 - D + \epsilon)} \right\}.$$

This being true for every $\epsilon > 0$, we have

$$\limsup_{\theta_n \to 0} u_M(r e^{i\theta_n}) \geq \frac{2}{\pi} \tan^{-1} \left\{ \frac{cD}{1 + c^2(1 - D)} \right\}.$$

In particular, if we choose $c = (1 - D)^{-1/2}$,

$$\limsup_{\theta_n \to 0} u_M(r e^{i\theta_n}) \geq \frac{2}{\pi} \tan^{-1} \left\{ \frac{D}{2\sqrt{(1 - D)}} \right\}, \quad r e^{i\theta} \to 1, \quad \theta = c(1 - r).$$
This quantity is therefore a lower bound for \( u_M(h) \) for some \( h \in \mathcal{B}_c^+ \).

Therefore, by Lemma 2.2 it is also a lower bound for \( u_M(h) \) for some \( h \in \mathcal{B} \). Since \( u_M = 0 \) on \( \mathcal{B}^- \), it follows that for some homomorphism \( h \) in \( \mathcal{B}^+ \),

\[
u_M(h) \geq \frac{2}{\pi} \tan^{-1}\left\{ \frac{D}{2\sqrt{1-D}} \right\}
\]

as was to be proved.

If \( M \) is a measurable subset of \( C \), we let \( \tilde{M} = \{ h \in \mathcal{D}_1 : \chi_M(h) = 1 \} \).

We recall that the range of \( f \in L^\infty \) on \( \tilde{M} \) is precisely the collection of essential cluster values of \( f(e^{i\theta}) \) as \( e^{i\theta} \to 1 \) through \( M \). We also recall the result from [1] that for each \( h \in \mathcal{D} \), \( \mu_h(\tilde{M}) = u_M(h) \).

With these preliminaries we may now prove:

**Corollary 2.6.** Let \( M \) be a measurable subset of \( C^+ \). Then, \( d(M) = 1 \) if and only if \( \mu_h(\tilde{M}) = 1 \) for every \( h \in \mathcal{B}^+ \).

**Proof.** If \( d(M) = 1 \), we have from Lemma 2.4 that \( u_M(h) = 1 \). From the above remark it is immediate that \( \mu_h(\tilde{M}) = 1 \). On the other hand suppose \( d(M) < 1 \). Then, \( D(\sim M) > 0 \) and so by Lemma 2.5 there is an \( h \in \mathcal{B}^+ \) with

\[
u_{\sim M}(h) \geq \frac{2}{\pi} \tan^{-1}\left\{ \frac{D(\sim M)}{2\sqrt{1-D(\sim M)}} \right\} > 0.
\]

Since \( u_{\sim M} + u_M = 1 \), \( u_M(h) < 1 \) so \( \mu_h(\tilde{M}) < 1 \) for some \( h \in \mathcal{B}^+ \).

3. **Applications.** It is now an easy matter to obtain several results connecting the one-sided behavior of an \( L^\infty \) function on \( C \) at 1 with its boundary behavior at 1 from inside \( D \).

A function \( f \) on \( C \) is approximately continuous from above at 1 with value \( \alpha \) if for every \( \epsilon > 0 \), the density \( d(\{ e^{i\theta} : |f(e^{i\theta}) - \alpha| \leq \epsilon, 0 \leq \theta \leq \pi \}) \) equals one. The main theorem upon which the applications are based is

**Theorem 3.1.** Let \( f \in L^\infty \). Then \( f \) is approximately continuous from above at 1 with value \( \alpha \) if and only if \( f \) is identically \( \alpha \) on the support of the representing measure of each upper barely tangential homomorphism.

**Proof.** For each \( \epsilon > 0 \) let \( M_\epsilon = \{ e^{i\theta} : |f(e^{i\theta}) - \alpha| \leq \epsilon, 0 \leq \theta \leq \pi \} \). Then, \( f \) is approximately continuous from above at 1 with value \( \alpha \) if and only if \( d(M_\epsilon) = 1 \) for each \( \epsilon > 0 \) and only if, by Corollary 2.6, \( \mu_h(\tilde{M}_\epsilon) = 1 \) for every \( \epsilon > 0 \) and every \( h \in \mathcal{B}^+ \). This latter statement implies that for each \( h \in \mathcal{B}^+ \), the support of \( \mu_h \) is contained in \( \tilde{M}_\epsilon \) for
every \( \varepsilon \). But on \( \tilde{M}_e \), \( |f(h) - \alpha| \leq \varepsilon \). Thus for each \( h \in \mathbb{B}^+ \), \( f \) is identically \( \alpha \) on the support of \( \mu_h \). On the other hand suppose for each \( h \in \mathbb{B}^+ \) that \( f = \alpha \) on the support of \( \mu_h \). Then, since \( |f - \alpha| \geq \varepsilon \) on \((C - M)_e\) it must be that the support of \( \mu_h \) is entirely contained in \( \tilde{M}_e \), i.e., \( \mu_h(\tilde{M}_e) = 1 \). This completes the chain of implications and the theorem follows.

The next theorem is a generalization to \( L^\infty \) functions of a result of Tanaka [6, Theorem 5], for \( H^\infty \). By this time our proof is a considerable simplification of that given by Tanaka.

**Theorem 3.2.** Let \( f \in L^\infty \). Then, necessary and sufficient conditions for \( f \) to be approximately continuous from above at 1 with value \( \alpha \) are

(i) \( f(h) = \alpha \) for all \( h \in \mathbb{B}^+ \).

(ii) The set, \( \{ e^\# : |f(e^\#)| \leq |\alpha| + \varepsilon \} \), has density 1 at 1 from above for every \( \varepsilon > 0 \).

**Note.** In Tanaka's theorem condition (i) was the statement that \( f \) tends to \( \alpha \) radially. From Lemma 2.2 we see that this is equivalent to our (i) for \( H^\infty \) functions.

**Proof.** First suppose \( f \) is approximately continuous from above at 1 with value \( \alpha \). By Theorem 3.1, \( f = \alpha \) on the support of the representing measure for each \( h \in \mathbb{B}^+ \). Immediately, \( f = \alpha \) on \( \mathbb{B}^+ \). Condition (ii) is necessary because

\[
\{ e^\# : |f(e^\#) - \alpha| \leq \varepsilon \} \subset \{ e^\# : |f(e^\#)| \leq |\alpha| + \varepsilon \}.
\]

Next, suppose the conditions (i) and (ii) are satisfied. Let \( N_\varepsilon = \{ e^\# : |f(e^\#)| \leq |\alpha| + \varepsilon \} \). By Corollary 2.6 and condition (ii), \( N_\varepsilon \) contains the support of the representing measure of each upper barely tangential homomorphism for every \( \varepsilon > 0 \). Thus, for every \( \varepsilon > 0 \) we have \( |f| \leq |\alpha| + \varepsilon \) on each such support. Thus, \( |f| \leq |\alpha| \) on each such support. By condition (i) if \( h \in \mathbb{B}^+ \), then \( f(h) = \alpha \). But, \( f(h) \) is the integral average of values not exceeding \( \alpha \). Therefore, \( f \) must be identically \( \alpha \) on the support of \( \mu_h \). By Theorem 3.1, again, \( f \) is approximately continuous from above at 1 with value \( \alpha \).

That condition (i) is necessary is a Lindelöf-type theorem for \( L^\infty \). Using Lemma 2.3 we state this theorem more concretely. It should be noted that because of Lemma 2.2 this theorem generalizes the usual Lindelöf theorem for \( H^\infty \): If \( f \in H^\infty \) and is approximately continuous from above at 1 with value \( \alpha \), then \( f \) tends to \( \alpha \) radially.

**Theorem 3.3.** Let \( f \in L^\infty \). If \( f \) is approximately continuous from above at 1 with value \( \alpha \), then the upper barely tangential cluster set of \( f \) at 1 consists of the single point \( \alpha \).
References


University of California, Santa Barbara, California 93106