THE ALMOST FIXED POINT PROPERTY FOR HEREDITARILY UNICOHERENT CONTINUA

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Abstract. It is shown that a hereditarily unicoherent Hausdorff continuum $X$ has the almost fixed point property with respect to continuum valued mappings and finite coverings by subcontinua of $X$.

In [3], de Groot, de Vries, and Van der Walt defined the almost fixed point property as follows: let $X$ be a space, let $F$ be a collection of functions from $X$ into $X$, and $\Delta$ be a collection of coverings of $X$. Then $X$ has the almost fixed point property with respect to $F$ and $\Delta$ if for each function $f$ in $F$ and covering $\alpha$ in $\Delta$, there is an element $A$ of $\alpha$ such that $A$ meets $fA$. Although it was apparently intended that the members of $F$ be single valued mappings, in this paper we will stretch the definition a bit and allow the members of $F$ to be multivalued functions.

A continuum is a compact connected space. A continuum $X$ is unicoherent provided that if $X$ is the union of two subcontinua $A$ and $B$, then $A \cap B$ is a continuum. $X$ is hereditarily unicoherent if every subcontinuum of $X$ is unicoherent. A continuum valued function on a continuum $X$ is a multivalued function $f$ which assigns to each $x$ in $X$ a subset $f(x) \subseteq X$ such that $f(A)$ is a continuum for each subcontinuum $A$ of $X$. Here $fA = \bigcup \{ f(x) : x \in A \}$. A multivalued function on $X$ is upper semicontinuous if given a neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $fU \subseteq V$. An upper semicontinuous multivalued function on $X$ which sends points to continua is a continuum valued function in the sense of the above definition.

Hopf [4] proved that a locally connected unicoherent metric continuum has the almost fixed point property with respect to continuous mappings and finite closed coverings of order 2. In [3] it is shown that

1. $E^2$ has the almost fixed point property with respect to continuous mappings and finite coverings by convex open sets,
2. $\overline{E}^2$ has the almost fixed point property with respect to orientation preserving topological isometries and finite coverings by arcwise connected sets, and

Received by the editors February 2, 1970.

AMS 1968 subject classifications. Primary 5485; Secondary 5455.

Key words and phrases. Almost fixed point property, hereditarily unicoherent continuum, continuum valued function, Scherrer fixed point theorem.

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(3) a unicoherent space has the almost fixed point property with respect to continuous mappings and coverings consisting of three connected open sets.

In this paper it will be shown that a hereditarily unicoherent Hausdorff continuum has the almost fixed point property with respect to continuum valued mappings and finite subcovers by subcontinua of $X$.

In [5], Wallace generalized the Scherrer fixed point theorem by showing that an upper semicontinuous multivalued mapping of a tree has a fixed point (a tree is a locally connected hereditarily unicoherent Hausdorff continuum); the Theorem of this paper implies this result. As a matter of related interest, we remark that it is known that an arcwise connected hereditarily unicoherent Hausdorff continuum has the fixed point property. (See Young [6]. This result was first proved for metric spaces by Borsuk [1].) It is not known if hereditarily unicoherent, hereditarily decomposable continua have the fixed point property.

Let $\alpha$ be a collection of subsets of a space $X$, and $A, B \subseteq \alpha$. A *chain* in $\alpha$ from $A$ to $B$ is a sequence $A_1, A_2, \ldots, A_n$ in $\alpha$ such that $A_1 = A$, $A_n = B$, and each two consecutive sets of the sequences intersect. The chain is *simple* provided that terms that are not consecutive do not intersect. If $\alpha$ is a finite closed covering of a connected space and $A, B \subseteq \alpha$, then there is a chain in $\alpha$ from $A$ to $B$, and every such chain contains a simple chain from $A$ to $B$. A subset $C$ of $X$ *links* two subsets $A, B$ of $X$ if $C$ meets both $A$ and $B$.

Note that if $X$ is a hereditarily unicoherent Hausdorff continuum, then the intersection of an arbitrary collection of subcontinua of $X$ is again a continuum. In what follows, this result will be used frequently and without explicit mention.

**Lemma 1.** Let $X$ be a hereditarily unicoherent Hausdorff continuum and $A, B, C, D$ be subcontinua of $X$. If $A$ and $B$ are disjoint and are linked by both $C$ and $D$, then $C$ meets $D$.

**Proof.** Suppose that $C \cap D$ is empty. Then the intersection of the continuum $A \cup B \cup C$ with $D$ is $A \cap D \cup B \cap D$, both sets in the union being nonempty. But $A$ and $B$ are disjoint, so that $A \cup B \cup C \cup D$ is not unicoherent.

**Lemma 2.** A collection $\alpha$ of subcontinua of a hereditarily unicoherent Hausdorff continuum has the finite intersection property if and only if any two elements of $\alpha$ have a nonempty intersection.

**Proof.** See Lemma 1 of [2].
**Theorem.** A nonempty hereditarily unicoherent Hausdorff continuum \(X\) has the almost fixed point property with respect to continuum valued mappings and finite coverings by subcontinua of \(X\).

**Proof.** Let \(\alpha\) be a finite covering of the space \(X\) and \(f\) be a continuum valued mapping on \(X\). We must show that there is an \(A \subseteq \alpha\) which meets \(fA\). If \(\text{Card}(\alpha) = 1\), we are finished. We proceed by induction on \(\text{Card}(\alpha)\), and assume that the Theorem is true for all finite coverings \(\beta\) of \(X\) by subcontinua of \(X\) for which \(\text{Card}(\beta) < \text{Card}(\alpha)\). We assume that \(A\) and \(fA\) are disjoint for every \(A\) in \(\alpha\).

We prove that if \(A\) and \(B\) are two elements of \(\alpha\) which have a nonempty intersection, then either

1. \(A \cap fB \neq \emptyset\) and \(B \cap fA = \emptyset\), or
2. \(A \cap fB = \emptyset\) and \(B \cap fA \neq \emptyset\).

Indeed the covering \(\beta\) of \(X\) having for elements the continuum \(A \cup B\) and the continua of \(\alpha\) distinct from \(A\) or \(B\) has one less element than \(\alpha\). By the inductive hypothesis and the assumption that \(C\) and \(fC\) are disjoint for every \(C\) in \(\alpha\), we have

\[
(A \cup B) \cap f(A \cup B) \neq \emptyset.
\]

This means that

\[
(A \cap fB) \cup (B \cap fA) \neq \emptyset.
\]

Consequently, the nonempty continuum \((A \cap fB) \cup (B \cap fA)\) is the union of two disjoint closed sets \(A \cap fB\) and \(B \cap fA\), and so one of (1) or (2) must hold.

Next we show that if \(A_1, A_2, \ldots, A_n\) is a simple chain in \(\alpha\) such that \(A_n \cap fA_1 \neq \emptyset\), then \(fA_1\) meets \(A_2\). For by (1)–(2), either \(fA_1\) meets \(A_2\) or \(fA_2\) meets \(A_1\). Assume that \(A_1\) meets \(fA_2\). Then \(fA_1 \cup A_n \cup \ldots \cup A_3\) and \(A_1\) link the disjoint subcontinua \(A_2\) and \(fA_2\). Since

\[
A_1 \cap (fA_1 \cup A_n \cup \ldots \cup A_3) = \emptyset,
\]

this violates Lemma 1. Therefore \(A_1\) and \(fA_2\) are disjoint, and so \(A_2\) meets \(fA_1\).

We will now show that for each positive integer \(n\), there is a sequence \(A_1, \ldots, A_{n-1}, B_n\) of distinct elements of \(\alpha\) and a sequence \(\alpha_1, \ldots, \alpha_{n-1}\) of subcollections of \(\alpha\) such that

1. \(A_i \subseteq \alpha_i, i = 1, \ldots, n-1\).
2. \(\alpha_k = \{A \subseteq \alpha : A\ \text{links} \ A_{k-1} \text{ and } fA_{k-1}\}, k = 2, \ldots, n-1\).
3. \(\bigcup \{B : B \subseteq \alpha_i, i < n\} \cap fB_n = \emptyset\).
4. \(A_1, \ldots, A_{n-1}, B_n\) is a simple chain in \(\alpha\).

Choose an arbitrary nonempty element \(A_1\) of \(\alpha\) and let \(\alpha_1 = \{A_1\}\).
Assume sequences which satisfy (3)-(6) have been defined for some integer \( n \geq 1 \). Since \( B_n \) meets \( A_{n-1} \) and \( fB_n \) and \( A_{n-1} \) are disjoint, we deduce that \( B_n \) meets \( fA_{n-1} \). In accordance with (4) we define

\[
\alpha_n = \{ A \in \alpha : A \text{ links } A_{n-1} \text{ and } fA_{n-1} \}.
\]

If \( A, B \in \alpha_n \), the disjoint subcontinua \( A_{n-1} \) and \( fA_{n-1} \) are linked by \( A \) and \( B \) so that by Lemma 1, \( A \) meets \( B \). Then Lemma 2 implies that \( \alpha_n \cup \{ A_{n-1} \} \) has the finite intersection property. Consequently

\[
H = (\cap \{ A \in \alpha_n \}) \cap A_{n-1} \neq \emptyset.
\]

Choose an element \( C \) in \( \alpha \) which meets \( fH \). Since \( B_n \in \alpha_n \) and \( fH \subseteq fB_n \), we deduce from (5) that \( C \in \alpha_i \) for \( i \leq n \) (note that \( A \) and \( fH \) are disjoint for every \( A \in \alpha_n \) since \( A \cap fA \) is empty for every \( A \in \alpha_n \)). Let \( C_1, \ldots, C_m = C \) be a simple chain in \( \alpha \) of minimal length which joins \( C \) to an element \( C_1 \in \alpha_n \), i.e. any simple chain in \( \alpha \) from \( C \) to an element in \( \alpha_n \) has at least \( m \) elements.

We show that \( A_1, \ldots, A_{n-1}, C_1 \) is a simple chain in \( \alpha \). If \( B_n \) meets \( C_2 \), we may take \( B_n = C_1 \) and the desired result follows from (6). If \( B_n \) and \( C_2 \) are disjoint, then \( B_n, C_1, C_2, \ldots, C_m \) is a simple chain in \( \alpha \). Because \( fB_n \) meets \( C_m \), we conclude that \( fB_n \) also meets \( C_1 \). On the other hand, the disjoint subcontinua \( A_{n-1} \) and \( fA_{n-1} \) are linked by \( C_1 \) and \( fA_{n-2} \), so that Lemma 1 implies that \( C_1 \) meets \( fA_{n-2} \). It follows that if \( C_1 \) meets \( A_{n-2} \), then \( C_1 \) is an element of \( \alpha_{n-1} \). But then because of (5), \( C_1 \) could not meet \( fB_n \), and this is a contradiction. Thus \( C_1 \) does not meet \( A_{n-2} \). Suppose now that \( C_1 \) meets \( A_i \) for some \( i < n-2 \). Let \( r \) be the largest integer such that \( A_r \cap C_1 \) is not empty; \( r \) is less than \( n-2 \). Then the disjoint subcontinua \( C_1 \) and \( A_{r+1} \cup \cdots \cup A_{n-2} \) link the disjoint subcontinua \( A_r \) and \( A_{n-1} \), which contradicts Lemma 1. This shows that \( A_1, \ldots, A_{n-1}, C_1 \) is a simple chain.

We next show that \( A_1, \ldots, A_{n-1}, C_1, C_2 \) is a simple chain. To this end assume that \( C_2 \) meets \( A_{n-1} \). Let \( k \) be the largest integer for which \( A_{n-1} \cap C_k \) is not empty; then \( k \geq 2 \), and \( A_{n-1}, C_k, C_{k+1}, \ldots, C_m \) is a simple chain from \( A_{n-1} \) to \( C_m \). Since \( fA_{n-1} \) meets \( C_m \), we conclude that \( fA_{n-1} \) also meets \( C_k \), from which it follows that \( C_k \) is in \( \alpha_n \). But then \( C_k, C_{k+1}, \ldots, C_m \) is a simple chain from \( C_m = C \) to an element \( C_k \) in \( \alpha_n \) of length less than \( m \), and we have a contradiction. Thus \( C_2 \) and \( A_{n-1} \) are disjoint, and as above, we can prove that \( A_1, \ldots, A_{n-1}, C_1, C_2 \) is a simple chain in \( \alpha \).

Define \( A_n = C_1 \). Note that since \( A_n, C_2, \ldots, C_m \) is a simple chain and \( C_m \) meets \( fA_n \), we have

\[
(7) \quad fA_n \cap C_2 \neq \emptyset.
\]
Let $B \in \alpha_n$. We prove that

\[(8) \quad fC_2 \cap B = \emptyset.\]

If $B$ does not meet $C_2$, then $B, A_n, C_2$ is a simple chain from $B$ to $C_2$. Hence if we assume that $fC_2$ meets $B$, we would conclude that $fC_2$ meets $A_n$, and (7) could not hold. This means that if $B$ and $C_2$ are disjoint, then $fC_2$ does not meet $B$. Now assume that $B$ meets $C_2$. If in addition $B$ meets $C_j$ for some $j > 2$, then $B, C_j, \ldots, C_m$ would contain a simple chain from $C_m = C$ to an element $B$ in $\alpha_n$ of length less than $m$, and this is not possible. Thus $B, C_2, \ldots, C_m$ is a simple chain. Since

\[fB \cap C_m \supset fH \cap C_m \neq \emptyset,\]

$fB$ meets $C_2$. Then (1) and (2) imply that $fC_2$ does not meet $B$. This proves (8).

Let $B \in \alpha_i$ for $i < n$. We prove that

\[(9) \quad fC_2 \cap B = \emptyset.\]

If $B$ does not meet $C_2$, let $r$ be the largest integer for which $B \cap A_r$ is not empty; then $r \leq n$ and $B, A_r, \ldots, A_n, C_2$ is a simple chain. If we assume that $B$ meets $fC_2$, we conclude that $fC_2$ meets $A_n$, and this violates (8). This means that (9) holds when $B$ and $C_2$ are disjoint. If $B$ meets $C_2$, then from $B \in \alpha_i$ we also have that $B$ meets $A_i$, and it is not difficult to see that $B$ meets $A_r$ for $i \leq r \leq n$. Suppose that $fC_2 \cap B$ is not empty. Then the disjoint subcontinua $C_2$ and $fC_2$ are linked by $B$ and $fA_n$, and Lemma 1 implies that $B$ meets $fA_n$. Consequently the disjoint subcontinua $A_n$ and $fA_n$ are linked by $B$ and $fA_{n-1}$ which implies that $B \cap fA_{n-1}$ is not empty. But then $B$ links $A_{n-1}$ and $fA_{n-1}$ so that $B$ is in $\alpha_n$, and from what we have just shown, this implies that $fC_2$ and $B$ are disjoint. Thus (9) follows.

In summary, we have shown that $fC_2$ and $B$ are disjoint for every $B \in \alpha_i$ for $i \leq n$. We now define $B_{n+1} = C_2$ and observe that $A_1, \ldots, A_n, B_{n+1}, \alpha_1, \ldots, \alpha_n$ satisfy (3)–(6) so that the proof of the inductive step is completed.

Finally, we observe that the above inductive construction is impossible since $\alpha$ is finite, and so there exists an $A$ in $\alpha$ which meets $fA$. This completes the proof of the Theorem.

We remark that all the hypotheses of the Theorem were used except for compactness, which is inherent in the term "Continuum." We call a space $X$ hereditarily unicoherent if any two closed connected subsets of $X$ have connected intersection. Then an exact analogue of the above Theorem can be proved for hereditarily uni-
coherent spaces provided one talks about closed connected subsets rather than subcontinua. However, in reality this would amount merely to placing a new interpretation on the word continuum, and so no significant generalization of our Theorem would be obtained.

The authors are indebted to the referee for his helpfulness in improving the exposition in this paper.

REFERENCES


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