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A GENERALIZATION OF A THEOREM OF POINCARÉ

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ABSTRACT. Let G be a finitely generated Fuchsian group of the first kind. Let φ be a cusp form, and f a solution to $\theta_2 f=\varphi$, where θ_2 is the Schwarzian derivative. Then for every $A\in G$, there is a Möbius transformation $\chi(A)$ such that $f\circ A=\chi(A)\circ f$. We show that the homomorphism χ from G to Möbius transformations determines φ. The theorem for the special case where G is the covering group of a compact surface was first proved by Poincaré.

The purpose of this note is to show that the estimates obtained in [4] for solutions of the Schwarzian differential equation improve a result that appeared in [3] about determining projective structures on Riemann surfaces from their coordinate cohomology classes.

We shall establish the following

THEOREM. *Let G be a finitely generated Fuchsian group of the first kind acting on the upper half plane U. If (χ, f) and (χ, g) are deformations of G, then $f=g$, whenever the Schwarzians of f and g are cusp forms for G.*

We recall some definitions. The pair (χ, f) is a *deformation* of G if (1) χ is a homomorphism of G into the group of Möbius transformations, and (2) f is a meromorphic local homeomorphism from U onto an (open) subset of the extended complex plane, such that $f\circ A=\chi(A)\circ f$, $A\in G$.

If (χ, f) is a deformation, then

$$\varphi=\theta_2 f=(f''/f')'-\tfrac{1}{2}(f''/f')^2$$

is a holomorphic 2-form (automorphic form of weight -4) for G; that is,

$$\varphi(Az)A'(z)^2=\varphi(z),\qquad z\in U,\quad A\in G.$$

It is a cusp form, if

$$\sup\left\{(2\operatorname{Im} z)^2\left|\varphi(z)\right|;z\in U\right\}<\infty.$$

REMARK. For U/G compact the theorem has been proven in [3].

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In this case, for every deformation (χ, f) of G, we have that $\theta_2 f$ is a cusp form. For G the covering group of a compact surface, the result is classical and due to Poincaré (see, for example, Hawley and Schiffer [2]). For G fixed point free, the theorem states that the coordinate cohomology class determines uniquely a bounded projective structure. See Gunning [1] for details. By a *bounded* projective structure we mean a projective structure determined by a cusp form.

PROOF OF THEOREM. The proof follows [2] and was already known to Poincaré. (See also [3].) Let $f = f_1/f_2$ and $g = g_1/g_2$, where

$$f_2 = (f')^{-1/2}, \qquad f_1 = ff_2,$$
$$g_2 = (g')^{-1/2}, \qquad g_1 = gg_2.$$

Let $v(z) = (f_1g_2 - f_2g_1)(z)$, $z \in U$. Then it is well known (and easy to verify) that the holomorphic functions f_j $(j=1, 2)$ are linearly independent solutions of the second order linear differential equation

$$2y'' + \varphi y = 0 \tag{1}$$

with $\varphi = \theta_2 f$. Similarly the functions g_j $(j=1, 2)$ solve (1) with $\varphi = \theta_2 g$. If y_1 is a solution of (1), then so is $(y_1 \circ A)(A')^{-1/2}$ for every $A \in G$. Thus for $z \in U$, $A \in G$,

$$f_1(Az)A'(z)^{-1/2} = a_A f_1(z) + b_A f_2(z),$$
$$f_2(Az)A'(z)^{-1/2} = c_A f_1(z) + d_A f_2(z),$$

and

$$g_1(Az)A'(z)^{-1/2} = \alpha_A g_1(z) + \beta_A g_2(z),$$
$$g_2(Az)A'(z)^{-1/2} = \gamma_A g_1(z) + \delta_A g_2(z),$$

where a_A, b_A, c_A, d_A and α_A, β_A, γ_A, δ_A are constants that depend only on A. Also $a_Ad_A - b_Ac_A \neq 0 \neq \alpha_A\delta_A - \beta_A\gamma_A$, since the new solutions are again linearly independent. We can, of course, conclude more. Since for $A \in G$

$$f_1/f_2 \circ A = \chi(A) \circ f_1/f_2 \quad \text{and} \quad g_1/g_2 \circ A = \chi(A) \circ g_1/g_2,$$

we deduce that for all complex ζ,

$$\frac{a_A\zeta + b_A}{c_A\zeta + d_A} = \chi(A)(\zeta) = \frac{\alpha_A\zeta + \beta_A}{\gamma_A\zeta + \delta_A}.$$

Thus for some nonzero constant r_A,

$$a_A = r_A\alpha_A, \qquad b_A = r_A\beta_A, \qquad c_A = r_A\gamma_A, \qquad d_A = r_A\delta_A.$$

A simple calculation now shows that

$$v(Az)A'(z)^{-1} = r_A^{-1}(a_A d_A - b_A c_A)v(z).$$

Thus v is a holomorphic multiplicative (-1)-form. We must examine the behavior of v at the punctures of U/G. Let p be a puncture on U/G. It involves no loss of generality to assume that the puncture p is generated by the parabolic element $A_0(z)=z+2\pi$. (Replace G by $B \circ G \circ B^{-1}$ for a suitable conformal self map B of U.) By [**4**, Lemma 1] $\chi(A_0)$ is also parabolic. Replacing (χ, f) and (χ, g) by $(B\chi B^{-1}, B \circ f)$ and $(B\chi B^{-1}, B \circ g)$, where B is a Möbius transformation, we may assume that $\chi(A_0)(z)=z+2\pi b$, $b\neq 0$. Under these assumptions (see the proof of Lemma 1 in [**4**]), we have, for some integers k and l,

$$c_{A_0} = 0, \qquad d_{A_0} = (-1)^k, \qquad a_{A_0} = (-1)^k, \qquad b_{A_0} \neq 0,$$

$$\gamma_{A_0} = 0, \qquad \delta_{A_0} = (-1)^l, \qquad \alpha_{A_0} = (-1)^l, \qquad \beta_{A_0} \neq 0,$$

$$f_2(z) = O(1) = g_2(z), \qquad z \to \infty, \tag{2}$$

$$f_1(z) = O(|z|) = g_1(z), \qquad z \to \infty, \tag{3}$$

and $r_{A_0}^{-1}(a_{A_0}d_{A_0} - b_{A_0}c_{A_0}) = (-1)^{k-l}$. Thus, we conclude

$$e^{2\pi i\alpha}v(z+2\pi) = v(z), \qquad z \in U, \tag{4}$$

where $e^{2\pi i\alpha}=(-1)^{k-l}$, with $\alpha=0$ or $\alpha=1/2$. From (2) and (3) we conclude that

$$v(z) = O(|z|), \qquad z \to \infty. \tag{5}$$

From (4) it is easy to deduce that v has a Fourier series expansion of the form

$$v(z) = \sum_{n=\infty}^{\infty} a_n e^{i(n-\alpha)z}, \qquad z \in U.$$

Equation (5) implies that $a_n=0$ for $n<0$.

If $\alpha\neq 0$, we conclude $a_0=0$ as well. In particular v has a zero at ∞ of order $N-\alpha \geqq 0$ where N is the smallest integer n such that $a_n\neq 0$. (Note the order of the zero at a puncture need not be an integer.) It is known (see for example, [**5**, page 274] that for a nonzero multiplicative q-form φ that is meromorphic in U and at the punctures on U/G, we have

$$\deg \varphi = q(|\omega|/2\pi),$$

where $|\omega|$ is the Poincaré area of a fundamental domain ω for G in U. Since v is holomorphic, $|\omega|>0$, $q=-1$, we conclude that $v=0$.

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