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THE FOURIER TRANSFORM IS ONTO ONLY WHEN THE GROUP IS FINITE

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ABSTRACT. A well-known result of Henry Helson is used to prove that a locally compact abelian group is finite if and only if the Fourier transform is a surjective map.

Let G be a locally compact abelian group (LCAG) with dual group $\hat{G}$ (see [3] for definitions and basic facts). It has been proved by Segal in [4] and Rajagopalan in [2] that the Fourier transform $T:L^1(G)\to C_0(\hat{G})$ is onto if and only if G is finite. Segal's proof appeals to the principal structure theorem for a LCAG and involves proving the result for groups with special properties and then for their Cartesian products, whereas Rajagopalan uses a theorem of Kakutani and Birkhoff to show that if G is extremally disconnected, then G must be discrete.

In this note, we provide a short proof using a well-known result about Helson sets.

DEFINITION. A compact subset H of G (G not discrete) is a Helson set if every continuous function on H is the restriction of a Fourier transform.

LEMMA. *Let G be nondiscrete with Haar measure m. If H is a Helson set in G, then $m(H)=0$.*

PROOF. Let f be the characteristic function of H and $\sigma=f\,dm$. Then if $m(H)\neq 0$, it follows that σ is a nonzero bounded Borel measure on H, and $\hat{\sigma}\in C_0(\hat{G})$ since $f\in L^1(G)$. But this contradicts the main result of [1].

The referee has pointed out that citing Theorem 5.6.10 in [3, p. 119] for this result of Helson would be unsatisfactory, since the proof there depends on Theorem 4.6.8 in [3, p. 90], which is precisely the result we wish to prove:

THEOREM. *Let G be a LCAG with dual group $\hat{G}$. Then the Fourier transform $T:L^1(G)\to C_0(\hat{G})$ is onto if and only if G is finite.*

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PROOF. Assume T is onto. Let M be a compact subset of G with positive Haar measure and let g be a continuous function on M. Using the Tietze Extension Theorem, we may conclude that g is the restriction of a Fourier transform. Since M has positive measure, the lemma implies that the $\hat{G}$ must be discrete and therefore G is compact. Now T being an onto topological map (open mapping), the adjoint maps $L^1(\hat{G})$ onto $L^\infty(G)$, which contains $C(G)=C_0(G)$. But the adjoint is precisely the inverse Fourier transform, which maps into $C_0(G)$. Hence $L^\infty(G)=C_0(G)$ and therefore the inverse transform is onto. The same reasoning as earlier shows that G must also be discrete and thus finite.

The converse is obvious.

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