METRIZABILITY OF LOCALLY COMPACT VECTOR SPACES

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Abstract. By use of the theory of characters and the Pontryagin-van Kampen theorem, it is shown that if $E$ is a locally compact vector space over a discrete division ring $K$ of characteristic zero and if $\dim_K E < 2^m$, where $m$ is the cardinality of $K$, then $E$ is metrizable.

The problem of determining whether a locally compact vector space over a discrete division ring is metrizable arises in the study of finite-dimensional locally compact vector spaces, because we have a fairly concrete picture of those that are metrizable: If $E$ is a finite-dimensional, metrizable, indiscrete locally compact vector space over a discrete field $K$ and if $\mathfrak{o}$ is the smallest open subspace of $E$, then the topological additive group $\mathfrak{o}$ admits the structure of finite-dimensional topological vector space over the locally compact field $F$, where $F$ is either the real field $\mathbb{R}$, the field $\mathbb{Q}_q$ of $q$-adic numbers, or the field $\mathbb{Z}_p((X))$ of power series over the field $\mathbb{Z}_p$ of integers modulo $p$, under a scalar multiplication satisfying $\alpha \cdot (\lambda x) = \lambda (\alpha \cdot x)$ for all $x \in E$, $\lambda \in K$, $\alpha \in F$; moreover, if $E$ is a topological algebra, then $\mathfrak{o}$ is an ideal and $\alpha \cdot (xy) = (\alpha \cdot x)y$, $\alpha \cdot (yx) = y(\alpha \cdot x)$ for all $\alpha \in F$, $x \in \mathfrak{o}$, $y \in A$; finally, $K$ is algebraically isomorphic to a subfield of finite codegree of a finite extension of $F$ [4, Theorems 3 and 5]. Here we shall consider the special case of this problem where the scalar field has characteristic zero.

First, we need a lower bound on the dimension of nonzero compact vector spaces. Let $K$ be a division ring, equipped with the discrete topology. We denote by $K^\hat{}$ the (compact) character group of the discrete additive group $K$, made into a right topological vector space over the field $\mathbb{Z}_p((X))$ of power series over the field $\mathbb{Z}_p$ of integers modulo $p$, under a scalar multiplication satisfying $\alpha \cdot (\lambda x) = \lambda (\alpha \cdot x)$ for all $x \in E$, $\lambda \in K$, $\alpha \in F$; moreover, if $E$ is a topological algebra, then $\mathfrak{o}$ is an ideal and $\alpha \cdot (xy) = (\alpha \cdot x)y$, $\alpha \cdot (yx) = y(\alpha \cdot x)$ for all $\alpha \in F$, $x \in \mathfrak{o}$, $y \in A$; finally, $K$ is algebraically isomorphic to a subfield of finite codegree of a finite extension of $F$ [4, Theorems 3 and 5]. Here we shall consider the special case of this problem where the scalar field has characteristic zero.

First, we need a lower bound on the dimension of nonzero compact vector spaces. Let $K$ be a division ring, equipped with the discrete topology. We denote by $K^\hat{}$ the (compact) character group of the discrete additive group $K$, made into a right topological vector space over $K$ by defining $u \cdot \lambda : x \mapsto u(\lambda x)$ for all $u \in K^\hat{}$, $\lambda \in K$, $x \in K$ [3, Theorem 1].

Theorem 1. If $K$ is an infinite division ring of cardinality $m$, then $\dim_K K^\hat{} = 2^m$.

Proof. Case 1. The characteristic of $K$ is zero. Then for some cardinal number $n$ the additive group of $K$ is isomorphic to $\mathbb{Q}^n$, the
direct sum of \( n \) copies of the additive group \( Q \) of rationals, where
\( n = m \) if \( m > \aleph_0 \) and where \( 1 \leq n \leq \aleph_0 \) if \( m = \aleph_0 \). Hence \( K^\sim \) is topologically
isomorphic to \((Q^\sim)^n\), the cartesian product of \( n \) copies of \( Q^\sim \) [2, (23.21), p. 364], and card \((Q^\sim)^n\) = \( c \) [2, (25.4), p. 404]. If \( m > \aleph_0 \), then
\( \text{card}(K^\sim) = c^n = 2^m > m \), so \( \dim_K K^\sim = 2^m \). If \( m = \aleph_0 \), then card \((Q^\sim)^n\) = \( c^n = c > m \), whence again \( \dim_K K^\sim = 2^m \).

Case 2. The characteristic of \( K \) is a prime \( p \). Then the additive
group of \( K \) is isomorphic to \( \mathbb{Z}_p^m \), so \( K^\sim \) is topologically isomorphic to
\((\mathbb{Z}_p)^m \). Hence card \((K^\sim)^n\) = \( p^m = 2^m > m \), so \( \dim_K K^\sim = 2^m \).

As a consequence of Theorem 1, we note that if \( K \) is uncountable, then \( K^\sim \) is a nonmetrizable compact \( K \)-vector space of dimension \( 2^m \) [3, Theorem 8].

**Theorem 2.** Let \( K \) be a discrete division ring of characteristic zero. If \( E \) is a locally compact, totally disconnected \( K \)-vector space, then \( E \) is metrizable.

**Proof.** Let \( Q \) be the prime field of \( K \). By [2, (7.7), p. 62], \( E \) contains a compact open subgroup \( V \). Let \( F = \bigcap \{ \alpha V : \alpha \in Q^* \} \). Then \( F \) is a compact vector space over \( Q \) and hence is connected [3, Theorem 9]. Thus \( F = (0) \). Hence, as \( V \) is compact, for any neighborhood \( W \) of zero there exist \( \alpha_1, \ldots, \alpha_n \in Q^* \) such that \( W \supseteq \alpha_1 V \cap \ldots \cap \alpha_n V \). Therefore \( \{ \alpha_1 V \cap \ldots \cap \alpha_n V : \alpha_1, \ldots, \alpha_n \in Q^* \} \) is a fundamental system of neighborhoods of zero in \( E \); in particular, \( E \) is metrizable.

**Theorem 3.** Let \( K \) be a discrete division ring of characteristic zero, and let \( m = \text{card}(K) \). If \( E \) is a locally compact \( K \)-vector space and if \( \dim_K E < 2^m \), then \( E \) is metrizable.

**Proof.** Let \( C \) be the connected component of zero. By Theorem 2, \( E/C \) is metrizable. By [2, (e), p. 47], it therefore suffices to show that \( C \) is metrizable. Hence we may assume that \( E \) is connected. By the theorem of Pontryagin and van Kampen [2, (9.14), p. 95], the topological additive group \( E \) is the topological direct sum of \( \mathbb{R}^n \) and \( H \), where \( H \) is a compact subgroup. Let \( u \) be the (continuous) projection of \( E \) on \( \mathbb{R}^n \) along \( H \). If \( h \in H \), then the closed additive subgroup \( (Zh)^- \) generated by \( h \) is compact as it is contained in \( H \); if \( \lambda \in K \), then \( (Zh)^- = \lambda(Zh)^- \), a compact subgroup, whence \( u((Zh)^-) = (0) \) as \( \mathbb{R}^n \) contains no nonzero compact additive subgroups, and therefore \( \lambda h \in (Z\lambda h)^- \subseteq H \). Hence \( H \) is a vector subspace of \( E \). By Theorem 1, [3, Theorem 6], and our hypothesis, \( H = (0) \). Hence \( E = \mathbb{R}^n \) and thus is metrizable.

If \( K \) is countable, we may improve Theorem 3:

**Theorem 4.** Assume the Continuum Hypothesis. If \( K \) is a countable, discrete division ring of characteristic zero and if \( E \) is a locally compact
K-vector space such that \( \dim_K E \leq c \), then \( E \) is metrizable.

**Proof.** As in the proof of Theorem 3, we may assume that \( E \) is the topological direct sum of \( R^n \) and a compact subspace \( H \). By [3, Theorem 6], \( H \) is topologically isomorphic to the compact \( K \)-vector space \( (K^\wedge)^n \), the cartesian product of \( n \) copies of \( K^\wedge \), for some cardinal number \( n \). If \( n > \aleph_0 \), then \( \text{card}(K^\wedge)^n = c^n > c \), so \( \dim_K E \geq \dim_K H > c \), a contradiction. Hence \( n \leq \aleph_0 \), so \( H \) is metrizable as \( K^\wedge \) is [3, Theorem 8]. Thus \( E \) is metrizable.

It is an open question whether similar theorems hold for locally compact vector spaces over fields of prime characteristic. At any rate, we may take care of the one-dimensional case:

**Theorem 5.** If \( E \) is an indiscrete, one-dimensional locally compact vector space over a discrete field \( K \), then there is a topology on \( K \) making \( K \) into an indiscrete locally compact field and \( E \) a topological vector space over \( K \), so topologized; in particular, \( E \) is metrizable.

**Proof.** The proof is similar to that of [3, Theorem 10]. We topologize \( K \) so that \( f: \lambda \mapsto \lambda a \) is a homeomorphism, where \( a \) is a nonzero vector. Then \( K \) is locally compact; \( (\lambda, \mu) \mapsto \lambda + \mu \) is continuous, since each of the maps \( (\lambda, \mu) \mapsto (\lambda a, \mu a) \mapsto \lambda a + \mu a = (\lambda + \mu)a \mapsto \lambda + \mu \) is; and for each \( \alpha \in \mathbb{K} \), \( \lambda \mapsto \alpha \lambda \) is continuous, since each of the maps \( \lambda \mapsto \lambda a \mapsto \alpha \lambda a \mapsto \alpha \lambda \) is. With the induced topology, the multiplicative group \( K^* \) satisfies the hypotheses of Ellis's theorem [1, Theorem 2], so \( K^* \) is a locally compact group. In particular, the mapping \( (\lambda, \mu) \mapsto \lambda \mu \) is continuous at \( (1, 1) \); it is therefore also continuous at \( (0,0) \), for if \( V \) is a neighborhood of zero, there exists a neighborhood \( U \) of zero such that \( (1 + U)(1 + U) \subseteq 1 + V \), whence \( UV \subseteq U + U + UU = (1 + U)(1 + U) - 1 \subseteq V \). Therefore \( K \) is an indiscrete locally compact field, so its topology is given by an absolute value; consequently, \( E \) is also metrizable. Clearly \( E \) is a topological vector space over \( K \), as each of the maps \( (\lambda, \mu a) \mapsto (\lambda, \mu) \mapsto \lambda \mu \mapsto \lambda \mu a \) is continuous.

**References**


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