ZERO DIVISORS AND NILPOTENT ELEMENTS
IN POWER SERIES RINGS

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ABSTRACT. It is well known that a polynomial $f(X)$ over a commutative ring $R$ with identity is nilpotent if and only if each coefficient of $f(X)$ is nilpotent; and that $f(X)$ is a zero divisor in $R[X]$ if and only if $f(X)$ is annihilated by a nonzero element of $R$. This paper considers the problem of determining when a power series $g(X)$ over $R$ is either nilpotent or a zero divisor in $R[[X]]$. If $R$ is Noetherian, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent; and $g(X)$ is a zero divisor in $R[[X]]$ if and only if $g(X)$ is annihilated by a nonzero element of $R$. If $R$ has positive characteristic, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent and there is an upper bound on the orders of nilpotency of the coefficients of $g(X)$. Examples illustrate, however, that in general $g(X)$ need not be nilpotent if there is an upper bound on the orders of nilpotency of the coefficients of $g(X)$, and that $g(X)$ may be a zero divisor in $R[[X]]$ while $g(X)$ has a unit coefficient.

1. Introduction. It is well known that a polynomial $f(X)$ over a commutative ring $R$ with identity is nilpotent if and only if each coefficient of $f(X)$ is nilpotent. In [1], McCoy establishes that a polynomial $f(X)$ is a zero divisor in $R[X]$ if and only if there is a nonzero element $r$ of $R$ with $rf(X) = 0$. In this paper, we consider the problem of determining when a power series $g(X)$ over $R$ is either nilpotent or a zero divisor in $R[[X]]$. We prove (Corollary 1) that if $R$ is Noetherian, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent. And if $R$ is Noetherian, then $g(X)$ is a zero divisor in $R[[X]]$ if and only if $g(X)$ is annihilated by some nonzero element of $R$ (Theorem 5). We establish (Theorem 1) that if $R$ has positive characteristic, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent and there is an upper bound on the orders of nilpotency of the coefficients of $g(X)$. We show by means of examples, however, that, in general, $g(X)$ need not be nilpotent if there is an upper bound on the orders of nilpotency of the coefficients.
of \( g(X) \), and that \( g(X) \) may be a zero divisor while \( g(X) \) has a unit coefficient.

Throughout this paper, \( R \) denotes a commutative ring with identity; \( \omega \) is the set of natural numbers; \( \omega_0 \) is the set of nonnegative integers; \( Z \) is the set of integers; and \( Q \) is the set of rational numbers. If \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \), we denote by \( A_f \) the ideal of \( R \) generated by the coefficients of \( f(X) : A_f = \{ f_0, f_1, f_2, \ldots \} R \). If \( A \) is an ideal of \( R \), we let \( A[[X]] = \{ f(X) = \sum_{i=0}^{\infty} f_i X^i : f_i \in A \text{ for each } i \in \omega_0 \} \) and we define \( A \cdot R[[X]] \) to be the ideal of \( R[[X]] \) which is generated by \( A \). Then \( A \cdot R[[X]] = \{ f(X) : A_i \subseteq B \} \) for some finitely generated ideal \( B \) of \( R \) with \( B \subseteq A \). It is clear that \( A \cdot R[[X]] \subseteq A[[X]] \); equality holds if and only if each countably generated ideal of \( R \) contained in \( A \) is contained in a finitely generated ideal contained in \( A \). In particular, if \( V \) is a valuation ring containing an ideal \( A \) which is countably generated but not finitely generated, then \( A \cdot V[[X]] \) is finitely generated. Finally, we note that if \( A \) is an ideal of \( R \), then \( R[[X]] / A[[X]] \cong (R/A)[[X]] \); hence \( A[[X]] \) is a prime ideal of \( R[[X]] \) if and only if \( A \) is a prime ideal of \( R \).

2. Nilpotent elements. Let \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \) and let \( n \in \omega_0 \); we define \( f^{(n)} = \sum_{i=0}^{n} f_i X^i \). Then \( f^{(n)} \) is zero or a polynomial of degree at most \( n \).

**Lemma 1.** Let \( R \) be a commutative ring with identity having characteristic a positive prime \( p \), and let \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \). The following conditions are equivalent:

(a) \( f(X) \) is nilpotent.
(b) There is a natural number \( m \) such that \( (f_i)^m = 0 \) for each \( i \in \omega_0 \).
(c) There is a natural number \( m \) such that \( (f^{(k)})^m = 0 \) for each \( k \in \omega_0 \).

**Proof.** (a) \( \leftrightarrow \) (b): This follows immediately from the fact that for each natural number \( n \), \( (f(X))^n = \sum_{i=0}^{\infty} (f_i)^n X^{np} \).

(b) \( \leftrightarrow \) (c): This is clear since for each natural number \( n \) and for each nonnegative integer \( k \), \( (f^{(k)})^n = \sum_{i=0}^{\infty} (f_i)^{np} X^{nk} \).

**Theorem 1.** Let \( R \) be a commutative ring with identity having positive characteristic \( n = p_1 p_2 \cdots p_t \) and let \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \). The following conditions are equivalent:

(a) \( f(X) \) is nilpotent.
(b) There is a natural number \( m \) such that \( (f_i)^m = 0 \) for each \( i \in \omega_0 \).
(c) There is a natural number \( m \) such that \( (f^{(k)})^m = 0 \) for each \( k \in \omega_0 \).

**Proof.** We let \( \phi_i : R[[X]] \to R[[X]] / p_i^1 R[[X]] \) be the natural homomorphism for \( 1 \leq i \leq t \). We note that for \( 1 \leq j \leq t \), \( R/p_j R \) has characteristic \( p_j \).
(a)\(\rightarrow\)(b): If \(f(X)\) is nilpotent, then for \(1 \leq j \leq t\), \(\phi_j(f(X))\) is nilpotent in \(R[[X]]/p_jR[[X]]\cong (R/p_jR)[[X]]\). By Lemma 1, there is, for \(1 \leq j \leq t\), a natural number \(m_j\) satisfying: For \(i \in \omega_0\), \(0 = (\phi_j(f_i))^{m_j} = \phi_j(f_i(m_j))\); that is, \((f_i)^{m_j} \in p_jR\) for each \(i \in \omega_0\).

Let \(m = m_1e_1 + m_2e_2 + \cdots + m te_t\). Then for each \(i \in \omega_0\),

\[
(f_i)^m = (f_i^{m_1})^{e_1}(f_i^{m_2})^{e_2} \cdots (f_i^{m_t})^{e_t} \in (p_1R)^{e_1}(p_2R)^{e_2} \cdots (p_tR)^{e_t} = (0).
\]

Hence (b) holds.

(b)\(\rightarrow\)(a): We assume that there is a natural number \(m\) satisfying:
\((f_i)^m = 0\) for each \(i \in \omega_0\). Then for each \(j\), \(1 \leq j \leq t\), \(\phi_j((f_i)^m) = (\phi_j(f_i))^{m_j} = 0\) for each \(i \in \omega_0\). By Lemma 1, there is for each \(j\), \(1 \leq j \leq t\), a natural number \(m_j\) satisfying

\[
[\phi_j(f(X))]^{m_j} = \phi_j((f(X))^{m_j}) = 0; \quad \text{that is, } (f(X))^{m_j} \in p_jR[[X]].
\]

Let \(m = m_1e_1 + m_2e_2 + \cdots + m te_t\); then

\[
(f(X))^m = [(f(X))^{m_1}]^{e_1}[(f(X))^{m_2}]^{e_2} \cdots [(f(X))^{m_t}]^{e_t}
\cdot (p_1R[[X]])^{e_1}(p_2R[[X]])^{e_2} \cdots (p_tR[[X]])^{e_t} = (0).
\]

Hence \(f(X)\) is nilpotent.

The proof that (b)\(\leftrightarrow\)(c) is analogous to the proof that (a)\(\leftrightarrow\)(b); hence it will be omitted.

**Theorem 2.** Let \(R\) be a commutative ring with identity and let \(f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]]\). We consider the following conditions:

(a) The ideal \(A_j\) is nilpotent.

(b) There is a natural number \(m\) which satisfies: \([A_j^{(0)}]^m = 0\) for each \(k \in \omega_0\).

(c) There is a natural number \(m\) which satisfies: \([f^{(k)}]^m = 0\) for each \(k \in \omega_0\).

(d) There is a natural number \(m\) which satisfies: \((f_i)^m = 0\) for each \(i \in \omega_0\).

(e) There is a natural number \(m\) which satisfies: \([f^{(k)}]^m \in (X^{k+1}) \cdot R[[X]]\) for each \(k \in \omega_0\).

(f) \((f(X))\) is nilpotent.

We have the implications (a)\(\leftrightarrow\)(b)\(\rightarrow\)(c)\(\rightarrow\)(e)\(\leftrightarrow\)(f) and (c)\(\rightarrow\)(d).

**Proof.**

(a)\(\rightarrow\)(b): For each \(k \in \omega_0\), \(A_j^{(0)} \subseteq A_j\); hence if \((A_j)^m = 0\), then \([A_j^{(0)}]^m = 0\) for each \(k \in \omega_0\).

(b)\(\rightarrow\)(a): Let \(m\) be a natural number which satisfies: \([A_j^{(0)}]^m = 0\) for each \(k \in \omega_0\). Let \(a \in (A_j)^m\); then for some \(i \in \omega_0\), \(a \in [A_j^{(i)}]^m = 0\). Thus \((A_j)^m = 0\).

(b)\(\rightarrow\)(c): Obvious.
(c)→(d): Let \( m \) be a natural number which satisfies: \([f^{(k)}]_{m} = 0\) for each \( k \in \omega_0 \). Then for each \( i \in \omega_0 \), \((f_i)^{m} = ([f^{(i)}]_{m})_{i} = 0\).

(c)→(e): Clear.

(e)→(f): We first observe that if \( i \leq k \), then for each \( m \in \omega \),

\[
([f(X)]^m)_i = \sum_{r_1 r_2 \cdots r_s = i}^{e_1 + e_2 + \cdots + e_s = m} \prod_{r_1 r_2 \cdots r_s = i} (n_1 n_2 \cdots n_s)
\]

where each \( n_1 n_2 \cdots n_s \in \omega \), implying that \( ([f(X)]^m)_i = ([f^{(k)}]_{m})_i \). For if \( r_1 e_1 + r_2 e_2 + \cdots + r_s e_s = i \) with each \( r_i \in \omega_0 \) and each \( e_i \in \omega \), then \( r_s \leq i \leq k \). Thus only the coefficients \( f_0, f_1, \cdots, f_s \), where \( t \leq k \), occur in the calculation of \( ([f(X)]^m)_i \), whereby we obtain the above equality.

Assuming (e), let \( m \) be a natural number which satisfies: \([f^{(k)}]_{m} \in (X^{k+1})R[[X]]\) for each \( k \in \omega_0 \). Then for each \( j \in \omega_0 \), \((f(X))^{m})_{j} = ([f^{(j)}]_{m})_{j} = 0\) since \([f^{(j)}]_{m} \in (X^{j+1})R[[X]]\). Hence \([f(X)]^m = 0\) and \( f(X) \) is nilpotent.

(f)→(e): We assume that \( [f(X)]^m = 0 \); then whenever \( i \leq k \), 0 = \([f(X)]^m)_i = ([f^{(k)}]_{m})_i \). Thus \([f^{(k)}]_{m} \in (X^{k+1})R[[X]]\) for each \( k \in \omega_0 \) and (e) holds.

Corollary 1. Let \( R \) be a commutative ring with identity and let
\( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \). If \( A_f \) is a finitely generated ideal of \( R \), then the following conditions are equivalent:

(a) The ideal \( A_f \) is nilpotent.

(c) There is a natural number \( m \) which satisfies: \([f^{(k)}]_{m} = 0\) for each \( k \in \omega_0 \).

(d) There is a natural number \( m \) which satisfies: \((f_i)^{m} = 0\) for each \( i \in \omega_0 \).

(f) \( f(X) \) is nilpotent.

(g) Each coefficient of \( f(X) \) is nilpotent.

Proof. In Theorem 2, we established the implications (c)→(d) and (a)→(c)→(f). That (d)→(g) is clear. Hence it suffices to prove that (f)→(g) and that (g)→(a).

(g)→(a): If (g) holds, then each element of \( A_f \) is nilpotent. Since \( A_f \) is finitely generated, \( A_f \) is nilpotent.

(f)→(g): Let \( \{P_\alpha\} \) be the collection of prime ideals of \( R \) and let \( N \) be the ideal of nilpotent elements of \( R \); then \( N = \cap P_\alpha \). For each \( \alpha \), \( P_\alpha[[X]] \) is a prime ideal of \( R[[X]] \). Since \( f(X) \) is nilpotent, \( f(X) \in P_\alpha[[X]] \) for each \( \alpha \). Hence \( f(X) \in \cap P_\alpha[[X]] = (\cap P_\alpha)[[X]] = N[[X]] \); that is, each coefficient of \( f(X) \) is nilpotent.
We now give examples which show that in Theorem 2, (c)$\leftrightarrow$(a) and (d)$\leftrightarrow$(f).

**Example 1.** Let $S = \mathbb{Z}/(p)$ where $p$ is a positive prime; let $\{ X_i \}_{i \in \omega_0}$ be a countable collection of indeterminates over $S$; and let

$$R = S[X_0, X_1, \ldots, X_n, \ldots]/\{ X_0^p, X_1^p, \ldots, X_n^p, \ldots \}.$$  

Let $f_i = X_i$ and let $f(X) = \sum_{i=0}^n f_i X^i \in R[[X]]$. Then for each $k \in \omega_0$, $[f^{(k)}]_p = 0$. But for each natural number $n$, $f_0 f_1 \cdots f_{n-1} \in (A_f)^n$ and $f_0 f_1 \cdots f_{n-1} \neq 0$. Thus $A_f$ is not nilpotent. We conclude that (c)$\not\leftrightarrow$(a).

**Example 2.** Let $n \in \omega$, $n \geq 2$, and let

$$R = Q[X_0, X_1, \ldots, X_n, \ldots]/\{ X_0^n, X_1^n, \ldots, X_n^n, \ldots \}.$$  

Let $f_i = X_i$ and let $f(X) = \sum_{i=0}^n f_i X^i \in R[[X]]$. It is clear that $(f_i)^n = 0$ for each $i \in \omega_0$; hence $f(X)$ satisfies (d).

We assume that $f(X)$ is nilpotent: $[f(X)]^m = 0$. Then $f_0^m = ([f(X)]^m)_0 = 0$ so $m \geq n$. Let $k_1$ be the smallest integer $l$ for which $f_0^n$ does not occur in every summand used in computing $([f(X)]^m)_l$. Then $0 = ([f(X)]^m)_{k_1} = a_0^{m-1} f_1^{m-(n-1)}$ plus other terms, each having $f_0^n$ as a factor, where $a \in \omega$. Hence $0 = ([f(X)]^m)_{k_1} = a_0^{m-1} f_1^{m-(n-1)}$, implying that $m - (n-1) \geq n$.

Let $k_2$ be the smallest integer $l$ for which some summand used in computing $([f(X)]^m)_l$ has neither $f_0^n$ nor $f_1^n$ as a factor. Then $0 = ([f(X)]^m)_{k_2} = b f_0^{m-1} f_1^{n-1} f_2^{m-2(n-1)}$ plus other terms, each having either $f_0^n$ or $f_1^n$ as a factor, where $b \in \omega$. Hence $0 = ([f(X)]^m)_{k_2} = b f_0^{m-1} f_1^{n-1} f_2^{m-2(n-1)}$, implying that $m - 2(n-1) \geq n$.

We can prove inductively by this process that for each $k \in \omega$, $m - k(n-1) \geq n$; that is, $m \geq n + k(n-1)$. This contradicts our assumption that $m \in \omega$, showing that $f(X)$ is not nilpotent. Hence (d)$\not\leftrightarrow$(f).

3. **Zero divisors.**

**Lemma 2.** Let $R$ be a commutative ring with identity and let $f(X) = \sum_{i=0}^n f_i X^i \in R[[X]]$. If for some natural number $t$, $f_i$ is regular in $R$ while $f_t$ is nilpotent for $0 \leq i \leq t-1$, then $f(X)$ is regular in $R[[X]]$.

**Proof.** We let $g(X) = \sum_{i=0}^{t-1} f_i X^i$ and $h(X) = \sum_{i=t}^n f_i X^i$; then $f(X) = g(X) + h(X)$. (We let $g(X) = 0$ if $t = 0$.) Since $g(X) = 0$ or $g(X)$
is a polynomial of which each coefficient is nilpotent, \( g(X) \) is nilpotent.

Let \( T \) denote the total quotient ring of \( R \) and let \( S = T[[X]] \) where \( M = \{ X^i \}_{i=1}^{\infty} \). Then in \( S \), we can write \( h(X) = X' h'(X) \) where \( h'(X) = \sum_{i=0}^{\infty} f_{i+1} X^i \); thus \( h(X) \) and \( h'(X) \) are associates in \( S \). Since \( f_i = (h'(X))_0 \) is regular in \( R \), \( f_i \) is a unit of \( T \), implying that \( h'(X) \) is a unit in \( T[[X]] \), hence also in \( S \). Since \( h(X) \) and \( h'(X) \) are associates in \( S \), \( h(X) \) is a unit in \( S \). Hence in \( S \), \( f(X) = g(X) + h(X) \) where \( g(X) \) is nilpotent and \( h(X) \) is a unit, implying that \( f(X) \) is a unit, hence is regular, in \( S \). Thus \( f(X) \) is regular in \( R[[X]] \).

**Theorem 3.** Let \( R \) be a commutative ring with identity in which each zero divisor is nilpotent, and let \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \). If some \( f_i \) is regular in \( R \), then \( f(X) \) is regular in \( R[[X]] \).

**Proof.** This is an immediate consequence of Lemma 2, letting \( t \) be the smallest integer \( k \) for which \( f_k \) is regular in \( R \).

**Corollary 2.** Let \( R \) be a commutative ring with identity in which each zero divisor is nilpotent. If the ideal \( N \) of nilpotent elements of \( R \) is nilpotent, then in \( R[[X]] \) each zero divisor is nilpotent.

**Proof.** Let \( f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] \) and assume that \( f(X) \) is not nilpotent. Then \( A_f \) is not nilpotent so \( A_f \subseteq N \); that is, not every coefficient of \( f(X) \) is nilpotent. By assumption, \( f(X) \) has a regular coefficient. By Theorem 3, \( f(X) \) is regular in \( R[[X]] \).

We observe that Corollary 2 can be restated as follows:

**Corollary 3.** Let \( R \) be a commutative ring with identity in which \( (0) \) is \( N \)-primary. If \( N \) is nilpotent, then \( (0) \) is a primary ideal of \( R[[X]] \).

We immediately have the following:

**Corollary 4.** Let \( R \) be a commutative ring with identity and let \( Q \) be a \( P \)-primary ideal of \( R \). If \( Q \supseteq P^k \) for some \( k \in \omega \), then \( Q[[X]] \) is a \( P[[X]] \)-primary ideal of \( R[[X]] \).

**Proof.** Since \( R[[X]]/Q[[X]] \cong (R/Q)[[X]] \), it follows from Corollary 3 that \( Q[[X]] \) is a primary ideal of \( R[[X]] \). Also, \( P^k \subseteq Q \) so that \( (P[[X]])^k \subseteq P^k [[X]] \subseteq Q[[X]] \); hence \( P[[X]] = \sqrt{P[[X]]} \subseteq \sqrt{Q[[X]]} \). And clearly \( \sqrt{Q[[X]]} \subseteq P[[X]] \). Hence \( \sqrt{Q[[X]]} = P[[X]] \) and \( Q[[X]] \) is \( P[[X]] \)-primary.

**Theorem 4.** Let \( R \) be a Noetherian ring with identity in which \( (0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) is a shortest primary representation, with \( \sqrt{Q_i} = P_i \). Then in \( R[[X]] \), \( (0) = Q_1[[X]] \cap Q_2[[X]] \cap \cdots \cap Q_n[[X]] \)
\[ \cap \cdots \cap Q_n[[X]] \text{ is a shortest primary representation with } \sqrt{Q_i[[X]]} = P_i[[X]]. \]

**Proof.** \( Q_i[[X]] \cap \cdots \cap Q_n[[X]] = (Q_i \cap \cdots \cap Q_n)[[X]] = (0). \) Further, Corollary 4 asserts that each \( Q_i[[X]] \) is \( P_i[[X]] \)-primary. It is straightforward to verify that this primary representation of \((0)\) in \( R[[X]] \) is, in fact, irredundant.

**Theorem 5.** Let \( R \) be a Noetherian ring with identity in which \((0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) is a shortest primary representation of \((0)\) with \( \sqrt{Q_i} = P_i, 1 \leq i \leq n. \) Then for \( f(X) = \sum_{i=0}^{\infty} f_iX^i \in R[[X]] \), these conditions are equivalent:

(a) \( f(X) \) is a zero divisor in \( R[[X]] \).

(b) \( f(X) \in P_i[[X]] \) for some \( i, 1 \leq i \leq n. \)

(c) There is a nonzero element \( r \) of \( R \) which satisfies \( rf(X) = 0. \)

**Proof.** (a)\(\rightarrow\) (b): This is an immediate consequence of Theorem 4 and \( [3, \text{Corollary 3, p. 214}]. \)

(b)\(\rightarrow\) (c): Assuming that \( f(X) \in P_i[[X]] \), this implies that \( A_f \subseteq P_i \).

Thus \((0) : A_f \neq (0) \) by \( [3, \text{Corollary 1, p. 214}]. \) Let \( r \in (0) : A_f, r \neq 0; \) then clearly \( r \in R \) and \( r \neq 0 \) while \( rf(X) = 0. \)

(c)\(\rightarrow\)(a): Obvious.

We conclude with an example which shows that Theorem 5 fails when \( R \) is not Noetherian.

**Example 3.**\(^2\) Let \( S \) be a commutative ring with identity; let \( \{Y, X_0, X_1, X_2, \ldots, X_t, \ldots\} \) be a set of indeterminates over \( S; \) and let

\[ R = S[Y, \{X_i\}_{i=0}^\infty] / (X_0 Y, \{X_i - X_{i+1} Y\}_{i=0}^\infty). \]

Let \( y = Y \) and let \( f(X) = y - X \). Then \( f(X) \) has a unit coefficient, so certainly \( rf(X) \neq 0 \) for each nonzero element \( r \) of \( R \). However, letting \( x_i = X_i \) and \( g(X) = \sum_{i=0}^\infty x_i X^i \), we see that \( f(X) \cdot g(X) = 0 \) while \( g(X) \neq 0. \)

**References**


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\(^{2}\) Example 3 was pointed out to the author by Professor Gilmer.