A GENERAL DIFFERENTIAL EQUATION FOR CLASSICAL POLYNOMIALS

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Abstract. Agrawal and Khanna [1] have derived the two partial
differential equations satisfied by the polynomial set \( B_n(x, y) \). In
this paper we shall present a generalization of these results.

Introduction. The purpose of the present paper is to derive three
partial differential equations satisfied by the polynomial set
\( W_{n;\gamma,\gamma'}^{\lambda,\lambda_1,\gamma_1} (u, v, x, y) \) which is the generalization of as many as forty
classical polynomials such as Legendre polynomials, Hermite poly-
nomials, Jacobi polynomials, Gegenbauer polynomials, Sister Celine
polynomials, Bedient polynomials, generalized Bessel polynomials
etc. The polynomial set \( W_{n;\gamma,\gamma'}^{\lambda,\lambda_1,\gamma_1} (u, v, x, y) \) has been defined by
means of the generating relation

\[
W_{n;\gamma,\gamma'}^{\lambda,\lambda_1,\gamma_1} (u, v, x, y) = \sum_{n=0}^{\infty} W_{n;\gamma,\gamma'}^{\lambda,\lambda_1,\gamma_1} (u, v, x, y) t^n,
\]

valid under the conditions given in [2]. Several other results for the
polynomial set \( W_{n;\gamma,\gamma'}^{\lambda,\lambda_1,\gamma_1} (u, v, x, y) \) have also been given in [2].

Substituting \( u^m \) for \( u \) and putting \( \gamma = 0, \gamma' = 0, \lambda = 0, \lambda' = 0 \) in
(1.1), we obtain [1, p. 646 (1.1)].

Differential equations for \( W_{n;\gamma,\gamma'}^{\lambda,\lambda_1,\gamma_1} (u, v, x, y) \). Expanding the left
hand side of (1.1) in ascending power of \( t \), using the equality
\[
\sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \psi(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \psi(k, n - mk)
\]
and equating coefficients of \( t^n \) on both sides, we have

\[
W_n = \sum_{z=0}^{n} \sum_{k=0}^{[n-z/m]} \sum_{\rho=0}^{[z/m]} \frac{[(a_p)_k]}{(a_{p'})_{s-m'_p}} \frac{(\gamma k + \lambda)_{n-s-mk}}{(s-m'_p)_{(n-s-mk)}} (\rho_{(s-m'_p)(n-s-mk)_{(s-m'_p)(k)}}(\rho_{k})) \times (\gamma' s - \gamma' m'_p + \lambda')_s{m_x}_{s-m'_p}{N'_y}_{s-m'_p}{N_u}_k{A v}_\rho,
\]

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Celine polynomials, Bedient polynomials, generalized Bessel polynomials.

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where \( W_n \) stands for \( W^\lambda,\lambda';m,m'(u, v, x, y) \).

Let us denote
\[
\theta_1 = x \frac{\partial}{\partial x}, \quad \theta_2 = y \frac{\partial}{\partial y}, \quad \theta_3 = u \frac{\partial}{\partial u} \quad \text{and} \quad \theta_4 = v \frac{\partial}{\partial v}.
\]

Let us consider
\[
\theta_2\{\theta_2 + (b'_D) - 1\} (\gamma'\theta_2 + \lambda' - \gamma'), \gamma' \{\gamma\theta_3 + \theta_1 + \lambda\} W_n.
\]

We have
\[
\theta_2\{\theta_2 + (b'_D) - 1\} (\gamma'\theta_2 + \lambda' - \gamma'), \gamma' \{\gamma\theta_3 + \theta_1 + \lambda\} W_n
\]
\[
= \sum_{s=1}^n \sum_{k=0}^{[n-s/m]} \sum_{\rho=0}^{[s/m']} \left\{ (s-m') \rho \right\} \left\{ s-m' \rho + (b'_D) \right\} (\gamma' \{ s-m' \rho \} + \lambda' - \gamma'), \gamma' \left\{ n+s-mk \right\} (\gamma + \gamma k + n-s-mk)
\]
\[
= \sum_{s=0}^{n-1} \sum_{k=0}^{[n-s-1/m]} \sum_{\rho=0}^{[s+1/m']} \left\{ (a_{p'}) + s-m' \rho \right\} (\gamma' \{ s-m' \ rho \} + \lambda')_{p+1}, \gamma' \left\{ n+s-mk \right\} (\gamma k + \lambda)_{n-s-mk}
\]
\[
\times \left\{ ((a_{p'}) + s-m' \rho) \gamma' \{ s-m' \ rho \} + \lambda' \right\}_{p+1}, \gamma' \left\{ n+s-mk \right\} (\gamma + \gamma k + n-s-mk)
\]
\[
= \{ N'y/mx \} \theta_1 \{ \theta_2 + (a_{p'}) \} (\gamma'\theta_2 + \lambda' + \theta_4)' \gamma'.
\]

Therefore,
\[
\begin{aligned}
mx \left\{ \theta_2 \prod_{i=1}^{q'} (\theta_2 + b'_i - 1)(\gamma'\theta_2 + \lambda' - \gamma'), \gamma' (\gamma\theta_3 + \theta_1 + \lambda) \right\} \\
- N'y \theta_1 \prod_{i=1}^{q'} (\theta_2 + a_i') (\gamma'\theta_2 + \lambda' + \theta_4) \right\} W_n
\end{aligned}
\]
\[
= 0,
\]

which is one of the differential equations for the polynomial set \( W^\lambda,\lambda';m,m'(u, v, x, y) \).

Similarly, it can be also shown that the other partial differential equations for \( W^\lambda,\lambda';m,m'(u, v, x, y) \) are given by

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\[
\left[ (N'\gamma)^m \theta_d \prod_{i=1}^{p'} (1 - a'_i - \theta_2 - m') \gamma (1 - \gamma' \theta_2 - \gamma'm') \gamma_m \right. \\
\left. \times (\theta_4 - \lambda' - \gamma' \theta_2 - \gamma'm') \gamma_m' - (-1)^{p'm'} \right.
\]
\[
\times (Av)(\gamma \theta_2 + \lambda' + \theta_4) (1 + \theta_2 - m') \gamma \prod_{i=1}^{q'} (b'_i + \theta_2 - m') \gamma_m' \right] W_n
\]
\[
= 0.
\]

and
\[
\left[ (mx)^m \theta_d \prod_{i=1}^{q} (\theta_3 + b_i - 1) (\lambda + \gamma \theta_3 - \gamma)(\lambda + \theta_1 + \gamma \theta_3 - \gamma) \right.
\]
\[
- N u (1 + \theta_1 - m) \gamma \prod_{i=1}^{p} (\theta_3 + a_i) \right] W_n
\]
\[
= 0.
\]

The equations (2.2), (2.3) and (2.4) are the partial differential equations satisfied by the polynomial set \( W_{n'} \gamma_m \).

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REFERENCES


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