Approximation by Homeomorphisms and Solution of P. Blass Problem on Pseudo-Isotopy

W. Holsztyński

Abstract. For every map \( f: S^n \to S^n \) of degree 1, existence of a pseudo-isotopy \( h: S^n \times I \to R = \{ z \in \mathbb{C} : |z| \geq 1 \} \) such that \( h(z, 0) = z \) and \( h(z, 1) = f(z) \) is established. On the other hand (i) maps of \( I^n \) into \( I^n \times 0 \subseteq E^{n+1} \) cannot be, in general, uniformly approximated by homeomorphic embeddings of \( I^n \) in \( E^{n+1} \) for \( n > 1 \), and (ii) maps of \( S^n \) into \( S^n \subseteq E^n \) of degree 1 cannot be, in general, extended to a pseudo-isotopy of \( S^n \) into \( E^{n+1} \).

P. Blass asked me: Can every mapping \( g: S^n \to S^n \) of degree 1 be obtained by a pseudo-isotopy in Euclidean space \( E^{n+1} \) from an embedding? Does there hold an analogous assertion for mappings of \( S^n \) into itself of other degree?

We will show that the answer is positive for \( n = 1 \) (see §1) and negative for \( n > 1 \) (see §4).

1. 1-dimensional case of a map of degree 1. First we will describe the most intuitive case. Some more general and stronger results are contained in §2.

Let \( C \) be the complex plane,

\[ S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \]
\[ R = \{ z \in \mathbb{C} : |z| \geq 1 \} \]

(1.1) Theorem. Let \( f: S^1 \to S^1 \) be a map of degree 1. Then there exists a homotopy \( F: S^1 \times I \to R \) such that \( F| S^1 \times \{ t \} \) is a homeomorphic embedding for every \( t \in I \setminus \{ 1 \} \), and \( F(z, 1) = f(z) \) for every \( z \in S^1 \).

Proof. Instead of pair \((R, S^1)\) we will consider the homeomorphic pair \((S^1 \times E^+, S^1 \times \{ 0 \})\), where \( E^+ \) is the set of all nonnegative reals. Let \( h: S^1 \times I \to S^1 \) be a homotopy such that \( h(z, 0) = 2 \) and \( h(2, 1) = f(z) \) for \( z \in S^1 \). Next, let

(1.2) \( u(z, t) = h(z, t)z^{-1} \) for \( (z, t) \in S^1 \times I \).

Then \( u(S^1 \times \{ 0 \}) = 1 \), so that there exists a mapping \( v: S^1 \times I \to E^+ \).
such that \( u = e^{2\pi i \cdot \varphi} \). Homotopy \( F_0: S^1 \times I \to S^1 \times E^+ \) given by formula \( F_0 = h \Delta v \) i.e. \( F_0(z, t) = (h(z, t), v(z, t)) \) has the following two properties: if \( i < 1 \) and \( z_1 \neq z_2 \) and \( h(z_1, t) = h(z_2, t) \) then \( u(z_1, t) \neq u(z_2, t) \) and \( F_0(z_1, t) \neq F_0(z_2, t) \). Thus \( F_0|_{S^1 \times \{t\}} \) is a homeomorphic embedding for \( t < 1 \). The second property is:
\[
F_0(z, 1) = (f(z), v(z, 1)).
\]
Thus the required homotopy \( F: S^1 \times I \to S^1 \times E^+ \) can be given by
\[
F(z, t) = \begin{cases} F_0(z, t/2) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ (f(z), (2 - 2t) \cdot v(z, 1)) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}
\]

**Remark.** A homotopy \( F \) on \( X \) such that \( F|_{X \times \{t\}} \) is a homeomorphic embedding for \( 0 \leq t < 1 \) is said to be a pseudo-isotopy (compare \( F \) from Theorem (1.1)).

2. 1-dimensional case of a map of degree \( n \). We will give a generalization of pseudo-isotopy.

(2.1) **Definition.** Given topological spaces \( X, Y \), a mapping \( g: X \to Y \) is isotopically dominated by a mapping \( f: X \to Y \) iff there exists a homotopy \( F: X \times I \to Y \) such that
\[
\begin{align*}
(i) & \quad F(x, 0) = f(x) \text{ and } F(x, 1) = g(x), \\
(ii) & \quad \text{if } 0 \leq t < 1 \text{ and } f(p) \neq f(q) \text{ then } F(p, t) \neq F(q, t) \text{ for every } p, q \in X, \\
(iii) & \quad \text{if } \frac{1}{2} \leq t < 1 \text{ then } F(p, t) = F(q, t) \text{ iff } F(p, \frac{1}{2}) = F(q, \frac{1}{2}), \text{ for every } p, q \in X.
\end{align*}
\]

The homotopy \( F \) will be called a pseudo-isotopy. In the case of a homeomorphic embedding \( f \) and compact \( X \) the homotopy \( F \) is a pseudo-isotopy in the usual sense.

Let us remark that in such a case also the homeomorphic embedding \( f \) is isotopically dominated by \( g \) (we can define a respective pseudo-isotopy \( G \) by \( G(x, t) = F(x, 1 - t) \)).

(2.2) **Theorem.** If, for \( f, g: S^1 \to S^1 \), \( R \ ord f = ord g \) then \( g \) is isotopically dominated by \( f \) in \( R \).

**Proof.** Let \( h: S^1 \times I \to S^1 \) be a homotopy that connects \( f \) and \( g \), i.e. \( h(z, 0) = f(z) \) and \( h(z, 1) = g(z) \). Next, let
\[
u(z, t) = h(z, t)/(f(z))^{-1} \quad \text{for} \quad (z, t) \in S^1 \times I \quad \text{(compare (1.2))}.
\]
Then \( u(S^1 \times \{0\}) = 1 \) so that there exists a mapping \( v: S^1 \times I \to E^1 \) into the real line \( E^1 \) such that \( u = e^{2\pi i \cdot v} \). Then the desired pseudo-isotopy \( F: S^1 \times I \to R \) is given by
\[
F(z, t) = v'(z, 2t) \cdot h(z, 2t) \\
= (2(1 - t)v'(z, t) + 2t - 1) \cdot g(z)
\]

for \(0 < t < \frac{1}{2}\),

\[
= (2(1 - t)v'(z, t) + 2t - 1) - g(z)
\]

for \(\frac{1}{2} \leq t \leq 1\),

where \(v'(z, t) = 1 + v(z, t) - \inf_{t \in \mathbb{R}} v(x, t)\).

Indeed, \(F\) is a well-defined mapping and condition (i) holds. Next, if \(f(p) \neq f(q)\) and \(h(p, t) = h(q, t)\) then \(u(p, t) \neq u(q, t)\), and consequently \(v'(p, t) \neq v'(q, t)\). But if \(h(p, t') \neq h(q, t')\) or \(v'(p, t') \neq v'(q, t')\) for \(t' = \min(2t, 1)\) and \(t < 1\), then \(F(p, t) \neq F(q, t)\). Thus condition (ii) holds. It is easy to see that condition (iii) also holds.

(2.4) Corollary. If \(g : S^1 \rightarrow S^1\) is a mapping of order 1 then there exists a homotopy \(F : S^1 \times I \rightarrow \mathbb{R}\) such that

\[
(* \quad F(z, 0) = z \quad \text{and} \quad F(z, 1) = g(z) \quad \text{for every} \quad z \in S^1
\]

\[
(** \quad F|S^1 \times \{t\} \quad \text{is a homeomorphic embedding for} \quad 0 \leq t < 1.
\]

Looking for \(F|S^1 \times \left[\frac{1}{2}; 1\right]\) at (2.3) it is easy to obtain the following

(2.5) Corollary. Let ord \(f = ord g \quad \text{for} \quad f, g : S^1 \rightarrow S^1\). Then there exist \(f_1 : S^1 \rightarrow S^1 \times I\) and \(f_2 : f_1(S^1) \rightarrow S^1\) such that \(f = f_2 \circ f_1\) and \(g = p \cdot f_1\), where \(p = S^1 \times I \rightarrow S^1\) is the projection \((p(z, t) = z)\).

(2.6) Corollary. If ord \(g = 1\) for \(g : S^1 \rightarrow S^1\), then there exists a homeomorphic embedding \(f_1 : S^1 \rightarrow S^1 \times I \rightarrow S^1\) such that \(g = p \circ f_1\).

3. Approximations by homeomorphisms. Let \(Q^n = \{x \in \mathbb{R}^n : |x| = 1\}\), \(S^{n-1} = Q^n\) and let \(\varphi : S^{n-1} \rightarrow S^{n-1} \times \{0\} \subset E^{n+1}\) be a continuous mapping. Next let \(X_1\) be a space obtained from \(Q^n\) by identification of points \(x, x'\) such that \(\varphi(x) = \varphi(x')\) and let \(X_2\) be a space obtained from \(S^{n-1} \times I\) by identification of points \((x, 0)\) and \((\varphi(x), 1)\). Then \(X_1, X_2\) are the compact metrizable spaces such that

\[
H_{n-1}(X_1) = \mathbb{Z}_k \quad \text{and} \quad H_{n-1}(X_2) = \mathbb{Z}_{k-1}
\]

where \(k = ord \varphi\) (we shall consider Čech homology theory).

(3.1) Theorem. Under the assumption \(|ord \varphi| > 1\), there does not exist a sequence of homeomorphic embeddings of \(Q^n\) into \(E^{n+1}\) which is uniformly convergent to a mapping \(g : Q^n \rightarrow E^{n+1}\), such that \(g(x) = (\varphi(x), 0)\) for \(x \in S^{n-1}\) and \(g^{-1}(S^{n-1} \times \{0\}) = S^{n-1}\).

Proof. Let \(f : Q^n \rightarrow E^{n+1}\) be a homeomorphic embedding of \(Q^n\) into \(E^{n+1}\) such that

\[
\epsilon = \epsilon(f) = \max_{x \in S^{n-1}} |f(x) - g(x)| < 1.1
\]

\footnote{In fact, we think that there does not exist such \(f\).}
Then we define \( h_f : X_1 \rightarrow E^{n+1} \) as follows

\[
h_f(x) = f \left( \frac{x}{1 - \varepsilon} \right) \quad \text{for} \quad |x| \leq 1 - \varepsilon
\]

\[
= \frac{1 - |x|}{\varepsilon} \cdot f \left( \frac{x}{|x|} \right) + \left( 1 - \frac{1 - |x|}{\varepsilon} \right) \cdot g(x).
\]

Now, if for a sequence \( f_1, f_2, \ldots, \varepsilon = \varepsilon(f_n) \rightarrow 0 \) then the mappings \( h_f(X_1) \rightarrow R^{n+1} \) are arbitrarily fine (i.e. under a metric in \( X_1 \) the mappings \( h_f \) are \( \delta_n \)-mappings with \( \delta_n \rightarrow 0 \)). For this reason \( H_{n-1}(h_f(X_1)) \) contains a cyclic element of order \( k = \text{ord } \varphi \), for an embedding \( f \) (to prove it see for instance [1, p. 39] and [2]). But \( h_f(X_1) \) is a subspace of \( E^{n+1} \). This contradiction shows the truth of the theorem.

(3.2) Theorem. Let \( g : S^{n-1} \times I \rightarrow E^{n+1} \) be a mapping such that \( g(x, 0) = (x, 0) \), \( g(x, 1) = (\varphi(x), 0) \) for every \( x \in S^{n-1} \), and \( g^{-1}(S^{n-1} \times \{0\}) = S^{n-1} \times \{0, 1\} \). Then, under the assumption \( |\text{ord } \varphi - 1| > 1 \), there does not exist a sequence of homeomorphic embeddings of \( S^{n-1} \times I \) into \( E^{n+1} \) which is uniformly convergent to \( g \).

Proof. Let \( f : S^{n-1} \times I \rightarrow E^{n+1} \) be a homeomorphic embedding such that

\[
\varepsilon = \varepsilon(f) = \max_{(x,t) \in S^{n-1} \times I} |f(x, t) - g(x, t)| < \frac{1}{3}.
\]

Then we define \( h_f : X_2 \rightarrow E^{n+1} \) as follows:

\[
h_f(x, t) = f(x, t) \quad \text{for} \quad \varepsilon \leq t \leq 1 - \varepsilon,
\]

\[
= \frac{t}{\varepsilon} f(x, t) + \left( 1 - \frac{t}{\varepsilon} \right) g(x, t) \quad \text{for} \quad 0 \leq t \leq \varepsilon,
\]

\[
= \frac{1 - t}{\varepsilon} f(x, t) + \frac{t - 1 + \varepsilon}{\varepsilon} g(x, t) \quad \text{for} \quad 1 - \varepsilon \leq t \leq 1.
\]

Now we can repeat the arguments from the proof of Theorem (3.1).

4. \( n \)-dimensional case, \( n \geq 2 \). Let \( S^n \) be the unit sphere of Euclidean space \( E^{n+1} = E^n \times E^1 \), and let \( g : S^n \rightarrow S^n \) be given by

\[
g(x, t) = (s \cdot x, \frac{2t + 1}{2}) \quad \text{for} \quad -1 \leq t \leq 0 \quad \text{and} \quad s = \frac{1 - (2t + 1)^2}{|x|},
\]

\[
= (s \cdot \varphi(x), 1 - 2 \min(t, 1 - t)) \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and} \quad s = 1 - 4(t - \frac{1}{2})^2,
\]
where \( \varphi : S^{n-1} \to S^{n-1} \) is a mapping of order \( \neq 0, 1 \). It is easy to see that \( \text{ord } g = 1 \) i.e. that \( g \) is homotopic to the identity mapping. But there does not exist a pseudo-isotopy for \( g \) i.e. such a homotopy \( F : S^n \times I \to E^{n+1} \) that \( F|S^n \times \{0\} \) is a homeomorphism for \( 0 \leq t < 1 \) and that \( F(x, 1) = g(x) \) for every \( x \in S^n \). Furthermore, let

\[
P_1 = \{(x, t) \in S^n : t \geq \frac{1}{2}\}, \quad P_2 = \{(x, t) \in S^n : t \leq \frac{1}{2}\}
\]

and \( g_i = g|P_i \) for \( i = 1, 2 \). We denote also by \( p : E^{n+1} \to E^n \) the projection given by \( p(x, t) = x \). Then the following lemmas hold; these are the consequences of the result of §3.

(4.1) **Lemma.** If \( \left| \text{ord } \varphi \right| > 1 \) then there does not exist a sequence of homeomorphic embeddings \( f_n : P_1 \to E^{n+1} \) which is uniformly convergent to \( g_1|P_1 \).

(4.2) **Lemma.** If \( \left| \text{ord } \varphi - 1 \right| > 1 \) then there does not exist a sequence of homeomorphic embeddings \( f_n : P_2 \to E^{n+1} \) which is uniformly convergent to \( g_2|P_2 \).

(4.3) **Corollary.** For every \( n \geq 2 \) there exists a mapping \( g : S^n \to S^n \subseteq E^{n+1} \) of order 1, that is not isotopically dominated by a homeomorphic embedding.

**References**


**University of Michigan, Ann Arbor, Michigan 48104.**