THE ORTHOMODULAR IDENTITY AND METRIC COMPLETENESS OF THE COORDINATIZING DIVISION RING

RONALD P. MORASH

Abstract. Let $F$ be any division subring of the real quaternions $H$. Let $l_2(F)$ denote the linear space of all square summable sequences from $F$ and let $L$ denote the lattice of all "⊥-closed" subspaces of $l_2(F)$, where "⊥" denotes the orthogonality relation derived from the $H$-valued form $(a, b) = \sum_{i=1}^{\infty} a_i b_i$ where $a, b \in l_2(F)$, $a = (a_i; i = 1, 2, \ldots)$ and $b = (b_i; i = 1, 2, \ldots)$. Then $L$ is complete, orthocomplemented, $M$-symmetric, irreducible, atomistic, and separable, but $L$ is orthomodular if and only if $F$ is either the reals, the complex numbers, or the quaternions.

The lattice of all closed subspaces of infinite-dimensional, separable, complex Hilbert space has these seven lattice-theoretic properties:

(i) complete [1, p. 6];
(ii) orthocomplemented [1, p. 52], [2, p. 42];
(iii) atomistic (Every element is the join of the atoms beneath it.) [3, p. 48];
(iv) irreducible (The center consists precisely of 0 and 1.) [1, p. 67], [3, p. 27];
(v) separable (An orthogonal family of atoms has at most countably many elements.) and infinite dimensional [3, p. 58];
(vi) $M$-symmetric (If $a, b \in L$, we write $a \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$. $L$ is $M$-symmetric if $a \leq b$ implies $b \leq a$.) [1, p. 82], [3, p. 2];
(vii) orthomodular (If $a, b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a')$.) [1, p. 53], [2, p. 42].

Real and quaternionic Hilbert space have the same properties. The question arises whether these are the only three lattices (up to ortho-isomorphism) having them. The problem underlying this question is one of coordinatization, that is, the realization of an abstract lattice, described only by algebraic properties, as a lattice associated in some natural way with a concrete object, for example, the lattice of projections of Hilbert space. Work towards a coordinatization theorem for lattices with the above properties has been done.
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by MacLaren [4], [5] and Zierler [6]. The former showed that if $L$ has properties (i) through (vi) and if dim $L \geq 4$, then $L$ is ortho-isomorphic to the lattice of closed subspaces of a semi-inner-product space over some division ring $D$. Our question is whether the assumption (vii) is enough to force $D$ to be either the reals, the complex numbers, or the quaternions. An answer of "yes" would characterize completely, in terms of lattice-theoretic properties, projection lattices of Hilbert space and would thus be of great importance in the study of the logical foundations of quantum mechanics [7, p. 71]. We present here some evidence in support of the possibility of an affirmative answer. Let $F$ be any division subring of the real quaternions $H$. We denote by $|x| = (a^2 + b^2 + c^2 + d^2)^{1/2}$ the norm and by $\bar{x} = a - bi - cj - dk$ the conjugate of $x = a + bi + cj + dk \in H$. The map $x \rightarrow \bar{x}$ is an involutory anti-automorphism of $H$. Consider $l_2(F)$, the linear space of square-summable (with respect to the above norm) sequences from $F$. Define a definite, Hermitian, conjugate-bilinear "form" on $l_2(F), (\ , \ ) : l_2(F) \times l_2(F) \rightarrow H$, by the rule $(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$, where $x, y \in l_2(F), x = (x_n; n = 1, 2, \cdots)$ and $y = (y_n; n = 1, \cdots)$. Note that this form is $H$-valued, but not necessarily $F$-valued. For each subset $M$ of $l_2(F)$, define $M^\perp = \{y \in l_2(F); (x, y) = 0 \text{ for each } x \in M\}$. Call a subspace $S$ of $l_2(F)$ closed in case $S = S^\perp$. The map $S \rightarrow S^\perp$ of the lattice $L$ of all subspaces of $l_2(F)$ into itself is a closure operator [8, p. 1518] and so the lattice $L$ of all closed subspaces is complete and orthocomplemented. It is also easily seen that this lattice is irreducible, atomistic, and separable. However:

**Theorem.** $L$ is orthomodular if and only if $F = R, C, or H$.

**Proof.** Only the "only if" part of the theorem needs proof. We give the proof for the case $F \subseteq R$ only (that is, $F \subseteq R$ and $L$ orthomodular imply $F = R$). The proofs of the other two cases (that is, $F \subseteq C$, but $F \not\subseteq R$ and $F \subseteq H$, but $F \not\subseteq C$) follow from the fact that sequential convergence in $C$ or $H$ can be characterized in terms of coordinate-wise convergence in $R$. Choose $\gamma \in R$. We shall show that, if $L$ is orthomodular, then necessarily $\gamma \in F$. Let $x_0 = 1$ and let $x_n = n/2^n, \ n = 1, 2, 3, \cdots$. Let $x = (x_n; n = 0, 1, \cdots)$. Let $z_0$ be the greatest integer less than or equal to $\gamma \sum_{n=0}^{\infty} x_n^2$. Let $z = (z_n; n = 0, 1, \cdots)$ where $z_0 z_1 \cdots$ is the binary expansion of $\gamma \sum_{n=0}^{\infty} x_n^2$. Thus $\sum_{n=1}^{\infty} z_n / 2^n = \gamma \sum_{n=0}^{\infty} x_n^2 - z_0$. Let $y_0 = z_0$ and let $y_n = z_n / n$ for $n = 1, 2, \cdots$. Let $y = (y_n; n = 0, 1, \cdots)$. Letting $a = \text{sp}(x)$ and $b = \text{sp}(y)$, $a$ and $b$ are distinct atoms in $L$ so that, by orthomodularity, $c = (a \lor b) \land a^\perp \not= 0$ [3, p. 291]. Necessarily, $c$ is an atom so that $c = \text{sp}(\tau x + y)$ for some $\tau \in F$. But $c \subseteq a^\perp$ so that $(\tau x + y)$
\[ \sum_{n=0}^{\infty} (\tau x_n + y_n) x_n = \tau \sum_{n=0}^{\infty} x_n^2 + \sum_{n=0}^{\infty} x_n y_n. \]

Hence,

\[ \tau = -\gamma \left( \sum_{n=0}^{\infty} x_n^2 \right) / \sum_{n=0}^{\infty} x_n = -\gamma. \]

Since \( \tau \in F \), we may conclude \( \gamma \in F \), as desired.

Added in proof. The fact that \((a \vee b) \wedge a^\perp \neq 0\) for distinct atoms \(a, b\) also follows from \(M\)-symmetry, so the theorem remains valid if we replace “orthomodular” by “\(M\)-symmetric.” Hence, this \(L\) is orthomodular if and only if it is \(M\)-symmetric. It follows also that the closure operation \(M \rightarrow M^\perp\) is Mackey [8, p. 1518] only when \(F = R, C,\) or \(H\).

References


University of Massachusetts, Amherst, Massachusetts 01002