ZERO SETS OF FUNCTIONS FROM NON-QUASI-ANALYTIC CLASSES

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Abstract. It is well known that any closed subset of the line is the zero set of a $C^\infty$-function. One can also specify the orders of the zeros at the isolated points. The present paper improves this result by replacing the class of $C^\infty$-functions by any non-quasi-analytic class of $C^\infty$-functions.

If $\{M_n\}_{n=0}^\infty$ is a sequence of positive numbers we let $C\{M_n\}$ denote the set of functions $f$ in $C^\infty(R)$ to which there correspond $B_f$ and $\beta_f$ satisfying

$$||f^{(n)}||_\infty \leq \beta_f B^n f M_n, \quad n = 0, 1, \ldots.$$  

The purpose of this paper is to prove the following:

**Theorem.** Let $\{M_n\}_{n=0}^\infty$ be a sequence of positive numbers such that $\sum_{n=1}^\infty M_{n-1}/M_n < \infty$. Let $E$ be a closed set in $R$ and let $S$ be a set consisting of at most countably many isolated points of $E$. Let $d$ be a function which assigns a positive integer to each point in $S$. Then there is a function $f$ in $C\{M_n\}$ with $\{x \in R : f(x) = 0\} = E$ and furthermore for every $s$ in $S$ the order of the zero of $f$ at $s$ is $d(s)$.

We let $S$ contain only isolated points since any limit point of the zero set of $f$ could not be a zero of finite order for $f$. The Denjoy-Carleman Theorem [2, p. 376] shows that a condition such as $\sum_{n=1}^\infty M_{n-1}/M_n < \infty$ is necessary to prevent $C\{M_n\}$ from being quasi-analytic.

We will repeatedly use the following theorem which can be found in [1, pp. 79–84] where it is credited to H. Bray:

**Theorem.** Assume $\{N_n\}_{n=0}^\infty$ is a sequence of positive numbers such that $N_0 = 1$ and $\sum_{n=1}^\infty \lambda_n < \infty$ where $\lambda_n = N_{n-1}/N_n$. Assume $g_0$ is a bounded measurable function on $R$ which vanishes outside a compact set. For $n = 1, 2, \ldots$ define $g_n$ on $R$ by

$$g_n(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x + t)dt.$$
Then \( \{g_n\} \) converges uniformly to a function \( g \) in \( C^\infty(R) \) with \( \|g^{(n)}\|_\infty \leq \|g\|_\infty N_n \) for \( n = 0, 1, \ldots \).

We will first obtain some functions which will be used in building the function of our theorem. We let \( \{s_n\} \) be a strictly increasing sequence of positive numbers satisfying: \( s_1 = 1 \), \( s_n \) tends to \( \infty \), and \( \sum_{n=1}^\infty M_{n-1}s_n/M_n < \infty \). For example we could take

\[
s_n = (\text{const}) \left( \sum_{k=n}^\infty \frac{M_{k-1}}{M_k} \right)^{-1/2} \quad \text{for } n > 1.
\]

We define \( \{N_n\} \) as follows: \( N_0 = 1 \) and \( N_n = M_n/(s_1 \cdots s_n) \) for \( n = 1, 2, \ldots \). Then \( \sum_{n=1}^\infty N_{n-1}/N_n = \sum_{n=1}^\infty M_{n-1}s_n/M_n < \infty \) and we let \( \lambda \) denote this sum. By applying Bray’s theorem to the function which is 1 on \((-\lambda, \lambda)\) and 0 elsewhere we obtain a function \( g \) in \( C^\infty(R) \) which satisfies:

(i) \( 0 \leq g \leq 1 \);
(ii) \( g > 0 \) on \((-2\lambda, 2\lambda)\) and 0 elsewhere; and
(iii) \( \|g^{(n)}\|_\infty \leq N_n \) for \( n = 0, 1, \ldots \).

Scaled translates of \( g \), i.e. functions of the form \( Ag(a(t-b)) \), will be used to define \( f \) in complementary intervals of \( E \) whose endpoints do not belong to \( S \).

In order to define \( f \) in a complementary interval of \( E \) which has at least one endpoint in \( S \) we will use the following:

**Lemma.** Let \( k \) be a positive integer. Then there are functions \( h_1(t, k) \) and \( h_2(t, k) \) in \( C^\infty(R) \) such that

(i) \( 0 \leq |h_i| \leq 1, \ i = 1, 2; \)
(ii) \( h_i \neq 0 \) on \((0, 4\lambda)\) and 0 elsewhere, \( i = 1, 2; \)
(iii) there is a number \( c > 0 \) such that \( h_1(t, k) = ct^k \) on \([0, \lambda]\) and \( h_2(t, k) = c(t-4\lambda)^k \) on \([3\lambda, 4\lambda]\); and
(iv) \( \|h_i^{(n)}\| \leq N_n \) for \( n = 0, 1, \ldots \) and \( i = 1, 2. \)

**Proof.** We first observe that if \( P(x) \) is a polynomial and \( \mu > 0 \) then

\[
\frac{1}{2\mu} \int_{-\mu}^{\mu} P(x + t)dt
\]

is a polynomial having the same leading term as \( P(x) \). If for each \( n \) we apply Bray’s process to the function which is \( x^n \) on \([-\lambda, 2\lambda]\) and 0 elsewhere, we obtain functions \( R_n(x) \) which on \([0, \lambda]\) are polynomials with leading term \( x^n \). Determining coefficients \( a_i \) such that on \([0, \lambda]\), \( R_k(x) + a_{k-1}R_{k-1}(x) + \cdots + a_0R_0(x) = x^k \), we obtain a polynomial \( x^k + a_{k-1}x^{k-1} + \cdots + a_0 = Q(x) \) such that applying Bray’s process to the function which is \( Q(x) \) on \([-\lambda, 2\lambda]\) and 0 elsewhere.
yields a function which on $[0, \lambda]$ is $x^k$. Let $c>0$ be sufficiently small that $|cQ(x)| \leq 1$ on $[-\lambda, 3\lambda]$. Let $h$ be the function in $C^\omega(R)$ obtained by applying Bray's process to the function which is $cQ(x)$ on $[-\lambda, 2\lambda]$, 1 on $[2\lambda, 3\lambda]$, and 0 elsewhere. We obtain $h_1$ from $h$ by changing the definition of $h$ to be 0 on $(-\infty, 0]$. $h_2$ is obtained in a similar way.

We will use scaled translates of $g$, $h_1$, $h_2$ to define $f$ in the complementary intervals of $E$. We now introduce a function which will be used as a factor to decrease these functions on small complementary intervals. We define $h>0$ on $(0, \infty)$ by $h(t) = 1$ for $t$ in $[s_i^{-1}, \infty)$ and $h(t) = (s_1 \cdots s_{n-1}) s_n^{-n+1}$ for $t$ in $[s_n^{-1}, s_{n-1}^{-1}]$, $n = 2, 3, \cdots$. There are exactly two properties of $h$ which we will use. If $k$ is a nonnegative integer, then

(1) $\lim_{t \to 0^+} h(t) t^{-k} = \lim_{n \to \infty} h(s_n^{-1}) s_n^k = 0$; and

(2) $\sup_{t>0} h(t) t^{-k} = \sup_{n>0} h(s_n^{-1}) s_n^k = s_1 s_2 \cdots s_k$.

We choose a function $\sigma$ on $E$ which is 0 on $E \setminus S$ and which takes the values +1 and -1 on $S$ in such a way that it possesses the following property: assume $s$ and $t$ are in $S$ and $s$ is the largest number in $S$ which is smaller than $t$; then if $d(t)$ is odd, $\sigma(s)$ and $\sigma(t)$ have opposite signs, while if $d(t)$ is even then $\sigma(s)$ and $\sigma(t)$ have the same sign. The function $\sigma$ will be used to insure that the function we are building does not vanish in any complementary interval of $E$.

We now define $f$. We treat the case where the complement of $E$ has no unbounded components since the other case requires only an easy modification. We let $f$ be 0 on $E$ and write the complement of $E$ as $U(a_n, b_n)$ where each $(a_n, b_n)$ is a component of the complement of $E$. On $(a_n, b_n)$ we define

$$f(t) = h(b_n - a_n) \left\{ g(4\lambda [b_n - a_n]^{-1} [t - (a_n + b_n)/2]) \right\} \cdot (1 - |\sigma(a_n)|) (1 - |\sigma(b_n)|)$$

$$+ \sigma(a_n) h_1(4\lambda [b_n - a_n]^{-1} [t - a_n], d(a_n))$$

$$+ \sigma(b_n) h_2(4\lambda [b_n - a_n]^{-1} [t - a_n], d(b_n)).$$

We let $D$ be the union of the complement of $E$, the interior of $E$, and the set of isolated points of $E$. Every point of $D$ has a neighborhood on which $f$ is $C^\infty$. Also one checks that for $s$ in $S$ the order of the zero of $f$ at $s$ is $d(s)$ as desired.

We will now show $f$ is continuous on $R$. For $t$ in a component interval of length $l$ of the complement of $E$ and $n = 0, 1, \cdots$ we have

(*) $|f^{(n)}(t)| \leq 2 h(l) l^{-n} (4\lambda)^n N_n$.
which tends to 0 with $l$. Continuity of $f$ off $D$ follows easily from (*) with $n=0$.

We next show that $f$ is differentiable and also that $f'(t)$ is 0 for $t$ in $E \setminus S$. It is easily seen that if $x<y$ then there is a point $t_{xy}$ in $(x, y) \cap D$ such that $f(x) - f(y) = f'(t_{xy})(x-y)$. Assume $t_0$ is in $E \setminus S$ and let $\epsilon>0$. Using (*) with $n=1$ we see there is a number $c>t_0$ such that $|f'(t)| < \epsilon$ for $t$ in $(t_0, c) \cap D$. Thus for $t_0<s<t<c$ we have

$$
|f(s) - f(t)|/(s-t) = |f'(t_{xy})| < \epsilon.
$$

Letting $s \to t_0$ we have $|f(t)/(t-t_0)| < \epsilon$ for $t_0<t<\epsilon$. Hence $\limsup_{t \to t_0} |f(t)/(t-t_0)| < \epsilon$. We conclude that $f'(t_0)$ exists and is 0.

Using (*) again one checks that $f'$ is continuous on $R$.

Letting $f'$ play the role of $f$ in the above argument, we see that $f''$ exists as a continuous function and is 0 in $E \setminus S$. Continuing in this manner we see that $f$ is in $C^\infty(R)$.

For $t$ in a component interval of length $l$ of the complement of $E$ we have

$$
|f^{(n)}(t)| \leq 2h(l)l^{-n}(4\lambda^2)N_n \leq 2(4\lambda)^n M_n.
$$

Hence $|f^{(n)}(t)| \leq 2(4\lambda)^n M_n$ on the dense set $D$ and thus $f$ is in $C\{M_n\}$.

As an example of an application of our theorem we give the following:

**Corollary.** Let $E$ and $E'$ be closed sets of real numbers. Then there is a continuous solution $u(x, t)$ to the heat equation, $u_{xx} = u_t$, in the $(x-t)$-plane satisfying $\{t: u(0, t) = 0\} = E$ and $\{t: u_x(0, t) = 0\} = E'$.

**Proof.** We define

$$
u(x, t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{g^{(n)}(t)x^{2n+1}}{(2n+1)!}
$$

where $f$ and $g$ are in $C\{\Gamma(3n/2)\}$ with appropriate zero sets.

**References**


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