A PRODUCT INTEGRAL REPRESENTATION 
FOR AN EVOLUTION SYSTEM

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Abstract. This paper provides a product integral representation for a nonlinear evolution system. The representation is valid for expansive systems and provides an analysis in the nonexpansive case which is different from ones previously discovered.

In [7], D. Rutledge obtains a product integral representation for a nonexpansive, nonlinear semigroup. In [6], Neuberger gets such a representation for expansive semigroups by first considering nonexpansive evolution systems. This paper obtains a product integral representation for an expansive evolution system $M$. In this development, it is not required that $\lim_{h\to 0} h^{-1}[M(h,0) - 1]P$ exist. As a corollary to Theorem 3, a statement equivalent to the statement that $M$ is nonexpansive is found.

Suppose that $\{G, +, | \cdot | \}$ is a complete, normed, Abelian group and that $S$ is the set of real numbers. If $f$ is a function from $S$ to $G$ and $a > b$, then denote the range of the restriction of $f$ to $[b, a]$ by $f([b, a])$. Also, the statement that $\{s_p\}_0^n$ is a subdivision of $\{a, b\}$ means that $s$ is a decreasing sequence with $s(0) = a$ and $s(n) = b$. The statement that $t$ is a refinement of the subdivision $s$ means that $t$ is a subdivision of $\{a, b\}$ and that there is an increasing sequence $u$ so that $s(p) = t(u(p))$ for $1 \leq p \leq n$. Finally, if $\{f_p\}_1^n$ is a sequence of functions from $G$ to $G$ and $g$ is in $G$, then

$$\left[ \prod_{p=1}^n f_p \right](g) = f_1(f_2(\cdots f_n(g))).$$

An evolution system on $G$ is a function $M$ with domain contained in $S \times S$ so that if $x \geq y$ then $M(x, y)$ is a function from $G$ to $G$ having the following properties:

1. if $x \geq y \geq z$ then $M(x, y)M(y, z) = M(x, z)$ and $M(x, x) = 1$, the identity function on $G$, and

2. if $t$ is a number and $P$ is in $G$ then the function $g$ given by $g(x) = M(x, t)P$, for all $x \geq t$, is continuous.

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In order to obtain a product integral representation for the evolution system $M$, two additional conditions are used:

(3) there is an increasing, continuous function $\beta$ and a subset $D$ of $G$ so that

(a) if $P$ is in $D$ and $x > y$ then $M(x, y)P$ is in $D$, and

(b) if $P$ is in $D$, $\varepsilon > 0$, $a > b$, and $Q$ is in $M([b, a], b)P$ then there is a positive number $\delta$ so that if $R$ is in $M([b, a], b)P$, $|Q - R| < \delta$, and $a \geq x \geq y \geq b$, then

$$| [M(x, y) - 1]R - [M(x, y) - 1]Q | \leq \exp(\beta(x) - \beta(y)) - 1 \cdot \varepsilon,$$

and

(4) there is a nondecreasing, continuous function $\alpha$ so that if $x > y$ and $\exp(\alpha(x) - \alpha(y)) < 2$, then $2 - M(x, y)$ has range all of $G$ and, if $P$ and $Q$ are in $G$ then

$$| 2 - \exp(\alpha(x) - \alpha(y)) | \cdot | P - Q | \leq | [2 - M(x, y)]P - [2 - M(x, y)]Q |.$$

Remark. It follows from condition (4) that if $\exp(\alpha(x) - \alpha(y)) < 2$, then $[2 - M(x, y)]^{-1}$ has domain all of $G$, and if $P$ and $Q$ are in $G$ then

$$| [2 - M(x, y)]^{-1}P - [2 - M(x, y)]^{-1}Q | \leq [2 - \exp(\alpha(x) - \alpha(y))]^{-1} | P - Q |.$$

In this paper, the following three theorems are proved.

**Theorem 1.** Suppose that $P$ is in $D$, $a > b$, and $M$ satisfies conditions (1)-(4). It follows that $M(a, b)P = \Pi [2 - M]^{-1}P$—in the sense that if $\varepsilon > 0$, then there is a subdivision $s$ of $\{a, b\}$ so that if $\{t_p\}$ is a refinement of $s$ then

$$| M(a, b)P - \Pi_{p=1}^{n} [2 - M(t_{p-1}, t_p)]^{-1}P | < \varepsilon.$$

**Theorem 2.** Suppose that $M$ satisfies conditions (1)-(4), if $x > y$ then $M(x, y)$ is continuous from $G$ to $G$, $D$ is dense in $G$, $a > b$, and $P$ is in $G$, it follows that $M(a, b)P = \Pi [2 - M]^{-1}P$.

**Theorem 3.** Suppose that $G$ is a Banach space, $M$ satisfies conditions (1)-(3). If $x > y$ then $M(x, y)$ is continuous from $G$ to $G$, $D$ is dense in $G$, and $\rho$ is a continuous, real valued function which is of bounded variation on each interval. These are equivalent:

(a) If $x > y$ and $P$ and $Q$ are in $G$ then

$$| M(x, y)P - M(x, y)Q | \leq \exp(\rho(x) - \rho(y)) \cdot | P - Q |.$$
(b) If \( x > y \) and \( \exp(\rho(x) - \rho(y)) < 2 \), then \( 2 - M(x, y) \) has range all of \( G \) and, if \( P \) and \( Q \) are in \( G \), then
\[
[2 - \exp(\rho(x) - \rho(y))] \cdot |P - Q| 
\leq |[2 - M(x, y)]P - [2 - M(x, y)]Q|.
\]

**Indication of Proofs.** The following inequality is important in what follows; it may be established after considering the polynomial \( P(z) = 1 - 2z^3 + z^4 \). It is labeled Lemma 1 for later reference.

**Lemma 1.** If \( x \) is a number and \( 1 \leq x \leq (1 + \sqrt{5})/2 \) then \( [2 - x]^{-1} \leq x^2 \).

In the definitions and lemmas which follow, suppose that \( M \) satisfies conditions (1)-(4), \( a > b \), and \( \epsilon > 0 \).

**Definition.** Define functions \( \delta \) and \( B \) as follows: if \( P \) is in \( D \) and \( a \leq z \leq b \) then \( \delta(z, P) \) is the largest number \( d \) not exceeding 1 so that if \( Q \) is in \( M([z, a], z) \), then
\[
|M(z, y) - 1|Q - |M(z, y) - 1|P < [\exp(\beta(x) - \beta(y)) - 1] \cdot \epsilon.
\]
Also, \( B(z, P) \) is the largest number \( u \) not exceeding \( a \) so that if \( u > v > z \) then \( |M(u, z)^{P} - P| < \delta(z, P) \).

**Remark.** Note that the existence of \( \delta \) follows from condition (3) and of \( B \) follows from condition (2).

**Lemma 2.** Suppose that \( P \) is in \( D \). If \( a \leq x \leq b \), \( \{t_{p}\}_{p}^{n} \) is a subdivision of \( \{B(x, P), x\} \), and \( j \) is an integer in \( [1, n] \), then
\[
|M(t_{j-1}, t_{j}) - 1|M(t_{j}, t_{n})P - |M(t_{j-1}, t_{j}) - 1|P < [\exp(\beta(t_{j-1}) - \beta(t_{j})) - 1] \cdot \epsilon.
\]

**Indication of Proof.** If \( \{t_{p}\}_{p}^{n} \) is a subdivision of \( \{B(x, P), x\} \) and \( j \) is an integer in \( [1, n] \) then \( x \leq t_{j} < B(x, P) \). Thus \( |M(t_{j}, x)^{P} - P| < \delta(x, P) \). Now, \( M(t_{j}, x)P \) is in \( M([x, a], x)P \), so if \( a \leq u \leq v \geq x \) then
\[
|M(u, v) - 1|M(t_{j}, x)P - |M(u, v) - 1|P < [\exp(\beta(u) - \beta(v)) - 1] \cdot \epsilon.
\]

**Lemma 3.** Suppose that \( P \) is in \( D \), \( \{t_{p}\}_{p}^{n} \) is an increasing sequence with values in \( [b, a] \) and limit \( z \). There is a positive integer \( N \) so that if \( n > N \) then \( B(t_{n}, M(t_{n}, b)P) \geq z \).

**Indication of Proof.** Suppose that \( P \) is in \( D \) and \( t \) is an infinite increasing sequence with values in \( [b, a] \) and limit \( z \). The fact that \( \{M(t_{p}, b)P\}_{p=0}^{n} \) converges in \( G \) and has limit \( M(z, b)P \) follows from
condition (2). Let $Q$ be $M(z, b)P$. Since $Q$ is in $M([b, a], b)P$, there is a number $d$ so that $0<d<1$ and, if $|R - Q| < d$ and $R$ is in $M([b, a], b)P$ and $a \geq x \geq y \geq b$, then

$$|M(x, y) - 1|Q - |M(x, y) - 1|R| \leq \exp(\beta(x) - \beta(y)) - 1 \cdot \epsilon/2.$$ 

Let $w$ be so that if $z \geq u \geq w$ then $|Q - M(u, b)P| < \delta/4$. Let $n$ be so that $t_n > w$. First, $\delta(t_n, M(t_n, b)P) \geq \delta/2$ because: suppose $R$ is in $M([t_n, a], b)P$ and $|R - M(t_n, b)P| < \delta/2$. Then $|R - Q| < d$ so that if $a \geq x \geq y \geq b$ then

$$|M(x, y) - 1|M(t_n, b)P - |M(x, y) - 1|R| \leq \exp(\beta(x) - \beta(y)) - 1 \cdot \epsilon/2 + \epsilon/2.$$ 

Finally, $B(t_n, M(t_n, b)P) \geq z$ because: suppose that $t_n \leq v \leq z$. Then

$$|M(v, t_n)M(t_n, b)P - M(t_n, b)P| \leq |M(v, t_n)M(t_n, b)P - Q| + |Q - M(t_n, b)P| \leq \delta/4 + \delta/4 \leq \delta(t_n, M(t_n, b)P).$$ 

**Lemma 4.** Suppose that $P$ is in $D$. There is a subdivision $u$ of $\{a, b\}$ so that if $\{\tau_i\}^n_0$ is a refinement of $u$ and $p$ is an integer in $[1, n]$ then

$$|M(t_{\tau_i-1}, t_p)P - M(t_{\tau_i-1}, t_p) - 1|M(t_p, b)P| \leq \exp(\beta(t_{\tau_i-1}) - \beta(t_p)) - 1 \cdot 2\epsilon.$$ 

**Indication of Proof.** Suppose that $P$ is in $D$. By the previous lemma, there is a subdivision $\{u_q\}_m^0$ of $\{a, b\}$ so that if $q$ is an integer in $[1, m]$ then $u_q = B(u_q, M(u_q, b)P)$. Let $\{t_p\}_n^0$ be a refinement of $u$ and $p$ be an integer in $[1, n]$. Let $q$ be an integer in $[1, m]$ so that $u_{q-1} \geq t_{p-1} > t_p \geq u_q$. Then $|M(t_{p-1}, b)P - M(u_q, b)P| < \delta(u_q, M(u_q, b)P)$ and $|M(t_p, b)P - M(u_q, b)P| < \delta(u_q, M(u_q, b)P)$. Hence, if $a \geq x \geq y \geq u_q$, then

$$|M(x, y) - 1|M(t_{p-1}, b)P - |M(x, y) - 1|M(t_p, b)P| \leq \exp(\beta(x) - \beta(y)) - 1 \cdot 2\epsilon.$$ 

**Indication of Proof of Theorem 1.** Suppose that $P$ is in $D$. Let $u$ be a subdivision of $\{a, b\}$ as indicated in Lemma 4, $\{s_p\}_n^0$ be a refinement of $u$ so that if $p$ is an integer in $[1, m]$ then $\exp(\alpha(s_{p-1}) - \alpha(s_p)) < (1 + \sqrt{5})/2$, and $\{t_p\}_n^0$ be a refinement of $s$. By Lemma 1, if $p$ is an integer in $[1, n]$ and $P$ and $Q$ are in $G$, then

$$|2 - M(t_{p-1}, t_p)|^{-1}P - |2 - M(t_{p-1}, t_p)|^{-1}Q| \leq \exp(2\alpha(t_{p-1}) - \alpha(t_p)) \cdot |P - Q|.$$ 

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\[ \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} P - M(a, b) P \]

\[ = \sum_{j=1}^{n} \left\{ \prod_{p=1}^{n+1-j} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} M(t_{n+1-j}, t_n) b P \right. \]

\[ - \prod_{p=1}^{n-j} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} M(t_{n-j}, t_n) b P \right\} \]

\[ \leq \sum_{j=i}^{n} \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \]

\[ \cdot \left| M(t_{n+1-j}, t_n) b P - [2 - M(t_{n-j}, t_{n+1-j})] M(t_{n-j}, t_n) b P \right| \]

\[ = \sum_{j=1}^{n} \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \]

\[ \cdot \left[ M(t_{n-j}, t_{n+1-j}) - 1 \right] M(t_{n-j}, t_n) b P \]

\[ - \left[ M(t_{n-j}, t_{n+1-j}) - 1 \right] M(t_{n+1-j}, t_n) b P \]

\[ \leq \sum_{j=1}^{n} \left[ \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot [\exp(\beta(t_{n-j}) - \beta(t_{n+1-j})) - 1] \cdot 2 \varepsilon \right. \]

\[ \leq \exp(2[\alpha(a) - \alpha(b)]) \cdot [\exp(\beta(a) - \beta(b)) - 1] \cdot 2 \varepsilon. \]

To see this last inequality, one should note Lemma 2.2 of [4].

**Indication of Proof of Theorem 2.** Suppose that \( P \) and \( Q \) are in \( G \), \( a > b \), and \( \{t_p\}^n_1 \) is a subdivision of \( \{a, b\} \) so that, if \( p \) is an integer in \([1, n] \), then \( [2 - M(t_{p-1}, t_p)]^{-1} \) has domain all of \( G \).

\[ M(a, b) P - \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} P \leq M(a, b) P - M(a, b) Q \]

\[ + \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} Q - \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} P \]

\[ + \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} Q - M(a, b) Q \right]. \]

Thus, if \( D \) is dense in \( G \) and \( M(a, b) \) is continuous from \( G \) to \( G \), it follows from Lemma 1 that \( M(a, b) P = a \prod_{p=1}^{b} [2 - M]^{-1} P \).

**Lemma 5.** If \( \rho \) is a continuous function from \( S \) to \( S \) and is of bounded variation on each interval of \( S \), \( a > b \), and \( \varepsilon > 0 \), then there is a subdivision \( s \) of \( \{a, b\} \) so that \( \{t_p\}^n_0 \) is a refinement of \( s \) then
\[ |\exp(\rho(a) - \rho(b)) - \prod_{p=1}^{n} [2 - \exp(\rho(t_{p-1}) - \rho(t_p))]^{-1} | < \epsilon. \]

**Indication of Proof.** Notice that if \( \rho \) is continuous and of bounded variation on each interval of \( S \), \( a > b \), and \( \{t_p\}_{0}^{n} \) is a subdivision of \( \{a, b\} \) so that, if \( p \) is an integer in \([1, n]\), then \( \exp(\rho(t_{p-1}) - \rho(t_p)) < 2 \) then

\[
\prod_{p=1}^{n} [2 - \exp(\rho(t_{p-1}) - \rho(t_p))]^{-1} \leq \prod_{p=1}^{n} \left[ 2 - \exp \left( \int_{t_p}^{t_{p-1}} |d\rho| \right) \right]^{-1}.
\]

With techniques similar to those used in the proof of Theorem 1, it can be shown that, if

\[
\exp \left( \int_{t_p}^{t_{p-1}} |d\rho| \right) < \frac{1 + \sqrt{5}}{2} \quad \text{for} \quad p = 1, 2, \ldots, n,
\]

then

\[
\left| \prod_{p=1}^{n} [2 - \exp(\rho(t_{p-1}) - \rho(t_p))]^{-1} - \exp(\rho(a) - \rho(b)) \right|
\leq \exp \left( 3 \int_{a}^{b} |d\rho| \right) \cdot \sum_{j=1}^{n} \left| \exp(\rho(t_{n-j}) - \rho(t_{n+1-j})) - 1 \right|^2.
\]

The conclusion of the lemma follows.

**Indication of Proof of Theorem 3.** Suppose that \( G \) is a Banach space and that \( \rho \) is a function from \( S \) to \( S \) which is continuous and of bounded variation on each interval of \( S \). Suppose also that \( x > y \) and that \( M(x, y) \) is a function from \( G \) to \( G \) having the property that if \( P \) and \( Q \) are in \( G \) then \( |M(x, y)P - M(x, y)Q| \leq \exp(\rho(x) - \rho(y)) |P - Q| < 2|P - Q| \). As in Lemma 1 of [5], let \( X \) be in \( G \) and \( K(Z) \) be \( .5[X + M(x, y)Z] \) for each \( Z \) in \( G \). Then \( K \) is a contraction mapping and there is only one member \( Z \) of \( G \) so that \( 2Z - M(x, y)Z = X \). Furthermore, if \( P \) and \( Q \) are in \( G \), then

\[
|Q - P| \leq .5 \left| 2 - M(x, y) \right| Q - \left[ 2 - M(x, y) \right] P
\]

\[
+ .5 \exp(\rho(x) - \rho(y)) \left| P - Q \right|.
\]

Consequently, in Theorem 3, statement (a) implies statement (b). Finally, with \( G \) and \( \rho \) as supposed above, if \( M \) satisfies conditions (1)–(3), \( D \) is dense in \( G \), statement (b) of Theorem 3 holds, and \( x > y \), then, by Theorem 2, \( M(x, y)P = \prod_{u} [2 - M]^{-1} P \) for each \( P \) in \( G \) and, by Lemma 5,
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\[ z \prod^{x}[2 - M]^{-1} P - x \prod^{y}[2 - M]^{-1} Q \leq \exp(\rho(x) - \rho(y)) \, |P - Q| . \]

This completes the proof of Theorem 3.

**Examples.**

**Example 1.** Let \( G \) be a Banach space and \( T \) be a one-parameter semigroup of nonlinear transformations on \( G \). That is, \( T \) is a function from \([0, \infty)\) to the set of continuous transformations from \( G \) to \( G \) which satisfies

1. \( T(x)T(y) = T(x+y) \) if \( x, y \geq 0 \),
2. if \( P \) is in \( G \) and \( g_\rho(x) = T(x)P \) for all \( x \) in \([0, \infty)\) then \( g_\rho \) is continuous and \( \lim_{x \to 0^+} g_\rho(x) = P \),
3. \( |T(x)P - T(x)Q| \leq |P - Q| \) if \( x \geq 0 \) and \( P \) and \( Q \) are in \( G \), and
4. there is a dense subset \( D \) of \( G \) such that if \( P \) is in \( D \) then \( g_\rho \) is continuous with domain \([0, \infty) \). By Theorem 2, if \( P \) is in \( D \) and \( x > 0 \), then \( T(x)P = \prod^{x}[2 - T(-dI)]^{-1}P \). Compare [5] and Theorem 2 of [7].

**Example 2.** Let \( f \) be an increasing function from the real numbers onto the real numbers so that \( f' \) is continuous and nonincreasing. Suppose also that \( g \) is increasing and continuous, and that, for \( x > y \) and \( P \) a real number,

\[ M(x, y)P = f(g(x) - g(y) + t \, |P|) . \]

\( M \) satisfies (1)–(4) but \( \lim_{h \to 0^+} h^{-1}[M(h, 0) - 1]P \) may not exist. Compare Example 2 of [8], Example 3.4 of [1], and Theorem A of [6].

**Example 3.** In case \( M \) satisfies conditions (1) and (2) and if \( P \) and \( Q \) are in \( G \) and \( x > y \), then

\[ |M(x, y) - 1||P - [M(x, y) - 1]Q| \leq [\exp(\beta(x) - \beta(y)) - 1]|P - Q| , \]

then, according to [2] and [3], each value of \( M \) has range all of \( G \) and is invertible. This paper provides an alternate method for obtaining \( M(x, y)^{-1} \).

**References**


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