KRONECKER FUNCTION RINGS AND FLAT $D[X]$-MODULES

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Abstract. Let $D$ be an integral domain with identity. Gilmer has recently shown that in order that a $v$-domain $D$ be a Prüfer $v$-multiplication ring, it is necessary and sufficient that $D^*$ be a quotient ring of $D[X]$, where $D^*$ is the Kronecker function ring of $D$ with respect to the $v$-operation. In this paper the authors prove that in the above theorem it is possible to replace "a quotient ring of $D[X]^*$" with "a flat $D[X]$-module." Moreover, it is shown that $D^*$ is the only Kronecker function ring of $D[X]$ which can ever be a flat $D[X]$-module.

In the sequel $D$ will denote an integral domain with identity and $K$ will denote its quotient field. Otherwise, our notation is essentially that of [1].

Let $I(D)$ denote the collection of all fractional ideals of $D$. The mapping $F \mapsto F_v$ of $I(D)$ into $I(D)$, where $F_v = (F^{-1})^{-1}$, is called the $v$-operation on $D$. The $v$-operation satisfies the properties of a *-operation, and if the $v$-operation is endlich arithmetisch brauchbar, then we call $D$ a $v$-domain and we denote by $D^*$ the Kronecker function ring of $D$ with respect to the $v$-operation (for a detailed treatment of *-operations, Kronecker function rings and the $v$-operation, see [1, Chapters 26 and 28]). If the set of $v$-ideals of finite type is a group under $v$-multiplication, then $D$ is said to be a Prüfer $v$-multiplication ring. Let $D$ be a $v$-domain. Then in [2] Gilmer proves that $D$ is a Prüfer $v$-multiplication ring if and only if $D^*$ is a quotient ring of $D[X]$. Thus, in case $D$ is a Prüfer $v$-multiplication ring, $D^*$ is a flat $D[X]$-module. The converse is also true, but in order to prove it we require some preliminary results.

Lemma 1. Let $D$ be an integral domain. If $Q$ is a prime ideal of $D[X]$ such that $(D[X])_Q$ is a valuation ring and if $(Q \cap D)D[X] \subseteq Q$, then $Q \cap D = (0)$.

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Proof. Since $D_{Q \cap D} = (D[X])_Q \cap K$, there is no loss of generality in assuming that $D$ is a valuation ring and that $Q \cap D$ is its maximal ideal. $Q$ is generated mod $(Q \cap D)D[X]$ by a monic polynomial $f$. If $y \in Q \cap D$, then since $(D[X])_Q$ is a valuation ring and since $f \in (Q \cap D)(D[X])_Q$, it follows that $y = fg/h$, where $g \in D[X]$ and $h \in D[X] - Q$. If $y \neq 0$, then $f$ divides $h$ in $K[X]$, whence $f$ divides $h$ in $D[X]$ by [1, 8.4].

The following result, due to Richman [3, Theorem 2], will be of use.

**Lemma 2.** Let $D_1$ be an overring of $D$—that is, $D \subseteq D_1 \subseteq K$. In order that $D_1$ be a flat $D$-module it is necessary and sufficient that $(D_1)_{M_1} = D_{M_1 \cap D}$ for each maximal ideal $M_1$ of $D_1$.

We are now able to sharpen the aforementioned result of Gilmer.

**Theorem 3.** Let $D$ be a v-domain and let $D^v$ be the Kronecker function ring of $D$ with respect to the v-operation. The following conditions are equivalent:

1. $D$ is a Prüfer v-multiplication ring.
2. $D^v$ is a quotient ring of $D[X]$.
3. Each valuation overring of $D^v$ is of the form $(D[X])_{p(X)}$ where $D_p$ is a valuation overring of $D$.
4. $D^v$ is a flat $D[X]$-module.

Proof. The equivalence of (1) and (2) is given in [2]. That (2) implies (3) is a direct consequence of Lemma 1 and that (3) implies (4) follows from Lemma 2. Therefore, we need only show that (4) implies (2). We claim that $D^v = (D[X])_S$, where $S = \{f \in D[X] | (A_f)_v = D \}$. (Here, $A_f$ denotes the ideal of $D$ generated by the coefficients of $f$.) Clearly, $D^v \supseteq (D[X])_S$. Let $A$ be an ideal of $D[X]$ such that $AAD^v = D^v$. Then there exist $f_1, \ldots, f_n \in A$ such that $(f_1, \ldots, f_n)D^v = D^v$. Set $m = \max_{1 \leq i \leq n} \deg(f_i) + 1$ and put $f(X) = f_1 + f_2 X^m + \cdots + f_n X^{(n-1)m}$. Then by [1, 26.7], $(A_f)_v = D$ and hence $A \cap S \neq \emptyset$. Therefore, if $M$ is a maximal ideal of $(D[X])_S$, then $MD^v \subseteq D^v$ and there exists a maximal ideal $M'$ of $D^v$ such that $M = M' \cap (D[X])_S$. The result follows from Lemma 2.

On the basis of Theorem 3, one is led to ask what Kronecker function rings are flat $D[X]$-modules. The answer is given by

**Corollary 4.** If $D^v$ is a Kronecker function ring of $D$ which is a flat $D[X]$-module, then $D^v = D^v$.

Proof. It follows from Lemma 1 and Lemma 2 that each valuation overring of $D^v$ is of the form $(D[X])_{p(X)}$, where $D_{p(X)}$ is a valuation
overring of \(D\). Therefore, \(D = D' \cap K = (\cap a(D[X])_{p_a[x]} \cap K = \cap aD_{p_a}\) and it follows from [1, 36.13] that \(D' = D^e\).

References


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