A NEW CHARACTERIZATION OF DEDEKIND DOMAINS

E. W. JOHNSON AND J. P. LEDIAEV

Abstract. In this note it is shown that a Noetherian ring $R$ is a Dedekind domain if every maximal ideal $M$ of $R$ satisfies the cancellation law: if $A$ and $B$ are nonzero ideals of $R$ and $MA = MB$, then $A = B$.

Let $R$ be a Noetherian domain (commutative with 1). And let $S$ be the semigroup of ideals of $R$ under multiplication. It is well known that $R$ is a Dedekind Domain if, and only if, every element $A \in S$ satisfies the cancellation law: if $B, C \in S$ and $A \neq 0$, then $AB = AC$ implies $B = C$. Since a Dedekind domain has the property that every ideal is a product of primes, however, it is natural to ask if the assumption that every ideal is cancellable is necessary. In this note we show that a Noetherian ring is a Dedekind domain if every maximal ideal is cancellable.

For an extensive bibliography on Dedekind domains we refer the reader to [1].

The main tool used in the following is the theorem, due to Samuel [2], that if $Q$ is an ideal primary for the maximal ideal of a local ring $R$, then for sufficiently large values of $n$, the length of $R/Q^n$ is a polynomial in $n$ of degree equal to the rank of $M$. We denote this polynomial by $p_Q(x)$.

We begin with the following:

Lemma. Let $R$ be a local ring in which the maximal ideal $M$ satisfies the cancellation law. Then either $M = 0$ or $M$ has rank 1.

Proof. Since $M$ satisfies the cancellation law, either $M = 0$ or $0 : M = 0$. In the second case, set $M = (a_1, \ldots, a_d)$ and let $p(x)$ be the polynomial $p_M(x+1) - p_M(x)$. Then for sufficiently large values of $n$, $p(n)$ is the length of the $R$-module $M^n/M^{n+1}$, which is also the number of elements in a minimal base for $M^n$. Now, for all $n \geq 1$, $M^{nd+n} = M^{nd}(a_1^n, \ldots, a_d^n)$, so, by cancellation, $M^n = (a_1^n, \ldots, a_d^n)$. Hence $p(n) \leq d$ for all sufficiently large $n$. Since $0 : M = 0$, it follows that $p(x)$ has degree 0, and therefore that $p_M(x)$ has degree 1. Hence $M$ has rank 1. Q.E.D.

Received by the editors February 16, 1970.


Key words and phrases. Dedekind domain, cancellation law.

Copyright © 1971, American Mathematical Society
Theorem. Let \( R \) be a Noetherian ring such that every maximal ideal satisfies the cancellation law. Then \( R \) is a Dedekind Domain.

Proof. Assume that \( R \) is not a field. It suffices to show that for every maximal ideal \( M, R_M \) is a regular local ring of altitude 1. To do this, fix \( M \) and set \( \overline{R} = R_M \). We adopt the notation that for any ideal \( A \) of \( R \), \( \overline{A} = AR_M \). Then \( \overline{A}M : \overline{M} \) \( = (AM : M)R_M = \overline{A} \), so the maximal ideal \( \overline{M} \) of the local ring \( \overline{R} \) is cancellable. Since \( \overline{M} \neq 0 \), \( \overline{M} \) has rank 1 by the Lemma. Clearly, \( \overline{M} \) is not a prime of 0 in \( \overline{R} \), so there exists an element \( a \in \overline{R} \) such that \( a \subseteq \overline{M}, a \notin \overline{M}^2 \), and \( a \) is not an element of any prime of 0 (see, for example, [3, p. 406]). Then the ideal \( (a) \) is primary for \( \overline{M} \), so there exists an integer \( k \) such that \( \overline{M}^k \subseteq (a) \) and \( \overline{M}^{k+1} \subseteq (a) \) (where \( \overline{M}^{k} = \overline{R} \) if \( k = 0 \)). Hence \( \overline{M}^{k+1} = \overline{M}^{k+1} \cap (a) = (\overline{M}^{k+1} : (a))(a) \); and therefore either \( \overline{M}^{k+1} \subseteq \overline{M}(a) \) or \( \overline{M}^{k+1} = (a) \). However, if \( \overline{M}^{k+1} \subseteq \overline{M}(a) \), then \( \overline{M}(a) = \overline{M}((\overline{M}^k + (a)) \) and \( (a) = \overline{M}^k + (a) \), which contradicts the choice of \( k \). Hence \( \overline{M}^{k+1} = (a) \), so by the choice of \( a, k = 0 \) and \( \overline{M} \) is principal. Since \( \overline{M} \) is not a prime of 0 in \( \overline{R} \), this completes the proof.

References


University of Iowa, Iowa City, Iowa 52240