OVERRINGS OF PRINCIPAL IDEAL DOMAINS

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Abstract. All rings between a (right and left) principal ideal domain R and its skewfield Q(R) of quotients are quotient rings of R with respect to Ore-systems in R.

In [4] R. Gilmer and J. Ohm investigate commutative integral domains with the QR-property, i.e. with the property that every ring between the ring R and its quotient field Q(R) is a ring of quotients of R with respect to some multiplicatively closed system in R. All Dedekind domains with torsion class group have the QR-property.

In this note we describe all overrings (i.e. rings between R and its skewfield Q(R) of quotients) of a principal right and left ideal domain R, as left quotient rings with respect to appropriate Ore-systems in R. Contrary to the commutative case, we show that the left quotient ring with respect to an irreducible element need not be local. Also, a counterexample shows that for a principal left ideal domain which is not a principal right ideal domain, overrings exist which are not left quotient rings with respect to some Ore-system. See [1] for similar results.

Let R be a principal left and right ideal domain. Then R is a unique factorization domain, in which every nonunit element 0 ≠ a can be written as a product of irreducible elements:

\[ a = p_1 \cdots p_n, \quad p_i \text{ irreducible for all } i. \]

If \( a = q_1 \cdots q_m \) is another irreducible factorization of \( a \), then \( n = m \) and there exists a permutation \( \pi \) of \( \{1, \cdots, n\} \) such that \( R/Rp_i \) is isomorphic to \( R/Rq_{\pi(i)} \) as an R-left module for every \( i \) ([5, p. 34]).

We say an element \( a \) is similar to \( b \) in R, if \( R/Ra \) is isomorphic to \( R/Rb \) as a left-R module. It is well known, [3, p. 316], that \( a \) and \( b \) are similar if and only if there exists an element \( c \) in R with \( Ra + Rc = R \) and \( Rar \cap Rc = Rbc \). Similarity is an equivalence relation and left-right symmetric.

For each \( a \neq 0 \) in R and every irreducible element \( p \) in R we define a nonnegative integer \( h_p(a) = n \), if \( n \) is the number of irreducible factors similar to \( p \) in any irreducible factorization of \( a \). It follows

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that \( h_p(a) = 0 \) for all \( p \) if and only if \( a \) is a unit in \( R \). For any set \( \Delta \) of irreducible elements in \( R \) we define the set \( S_{\Delta} = \{ a \in R : \exists h_p(a) = 0 \text{ for all } p \in \Delta \} \).

We repeat the following definition: A multiplicatively closed set \( M \) of regular elements of \( R \) is called a (left-) Ore-system if for elements \( m \) in \( M \) and \( r \) in \( R \) there exist elements \( m_1 \) in \( M \) and \( r_1 \) in \( R \) such that \( r_1 m = m_1 r \). If \( M \) is an Ore-system in \( R \), then there exists the ring of quotients of \( R \) with respect to \( M \) ([6, p. 7]). With this notation we obtain the following result:

**Theorem.** Let \( R \) be a left and right principal ideal domain. Then any set \( S_{\Delta} \) is an Ore-system and every ring between \( R \) and its skewfield of quotients \( Q(R) \) can be obtained as a ring of quotients of \( R \) with respect to some \( S_{\Delta} \).

**Proof.** First we show that every \( S_{\Delta} \) is an Ore-system. That \( S_{\Delta} \) is multiplicatively closed follows from the unique factorization theorem stated in the beginning. It remains to show that for \( s \in S_{\Delta} \) and \( r \) in \( R \) elements \( s_1 \) in \( S_{\Delta} \) and \( r_1 \) in \( R \) exist with

\[
S_1 r = r s_1.
\]

Let \( Rs + Rr = Rd, s = s'd, r = r'd, \) and it follows

\[
Rs' + Rr' = R \quad \text{for } s' \text{ in } S_{\Delta} \text{ and } r' \text{ in } R.
\]

Further, \( Rs' \cap Rr' = R s_1 r' \) and \( s_1 \) is a product of irreducible factors similar to the irreducible factors of \( s' \) and is therefore contained in \( S_{\Delta} \) ([2, p. 51]). It follows that \( s r' = r s' \) or \( s r = r s \) for \( s_1 \) in \( S_{\Delta} \) and \( r_1 \) in \( R \). This proves that \( S_{\Delta} \) is an Ore-system and there exists an overring \( S_{\Delta}^{-1}R = \{ s^{-1}a, s \in S, a \in R \} \) of \( R \).

Now let \( T \) be any ring between \( R \) and \( Q(R) \), \( a^{-1}b \) an element in \( T \). We may assume \( bR + aR = R \), and from \( bx + ay = 1 \), for \( x, y \) in \( R \), it follows that \( a^{-1} \) belongs to \( T \). Therefore, the inverse of every irreducible factor of \( a \) is contained in \( T \). It remains to show that for \( p_1 \) similar to the irreducible element \( p \) in \( R \) with \( p^{-1} \) in \( T \) we have that \( p_1^{-1} \) is contained in \( T \).

Since \( p_1 \) is similar to \( p \), there exists \( c \) in \( R \) with \( Rp \cap Rc = Rp_1 c \) and \( Rp + Rc = R \). The element \( cp^{-1} \) is contained in \( T \) and can be written in the form \( cp^{-1} = a^{-1}b \) for \( a \neq 0, b \) in \( R \). But then \( ac = bp \) in \( Rp \cap Rc \) and this means \( ac = rp_1 c \) or \( a = rp_1 \) for some element \( r \) in \( R \). From this it follows that \( p_1^{-1} \) is contained in \( T \). This proves that \( T = S_{\Delta}^{-1}R \) where \( \Delta \) is the set of those irreducible \( q \) in \( R \) such that \( q^{-1} \) is not contained in \( T \).

**Remark 1.** Unlike the commutative case, \( S_{\Delta}^{-1}R \) is in general not a
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local ring for Δ consisting of just one irreducible element \( p \). \((S_p^{-1}R\) is local if and only if \( h_p(a+b) \geq \min(h_p(a), h_p(b)) \) \[2, p. 57\].) For example, consider the irreducible element \( p=x+t \) in the principal left and right ideal domain \( R=\mathbb{Q}(t)[x, \sigma] = \{ \sum_{i=0}^{n} f_i(t)x^i, f_i(t) \in \mathbb{Q}(t) \} \), where \( \mathbb{Q}(t) \) is the field of rational functions in one variable over the field of rationals \( \sigma \) and \( \sigma \) is the automorphism of \( \mathbb{Q}(t) \) defined by \( \sigma(t) = t+1 \). Addition in \( R \) is defined componentwise and multiplication by \( xf(t) = f'(t)x \).

Now it is easy to see that \( x+g(t)^{-1}g(t+1)(t+1) \) for every element \( 0 \neq g(t) \) in \( \mathbb{Q}(t) \), is similar to the element \( x+t \) in \( R \). From this we conclude that for example \( S_p^{-1}Rp \) and \( S_p^{-1}Rp_1 \), with \( p_1=x+t-1(t+1)^2 \), are two different maximal left ideals of \( S_p^{-1}R \) for \( p=x+t \).

Remark 2. The theorem just proved cannot be extended to left principal ideal domains with maximum condition on right principal ideals, even though the rings \( S^{-1}R \), as defined above, exist in this case too. Consider \( \mathbb{Q}(t)[x, \tau] = R \) as in the previous remark, but \( \tau \) is now the monomorphism from \( \mathbb{Q}(t) \) into \( \mathbb{Q}(t) \) defined by \( \tau(x) = x^2 \). \( R \) is a principal left ideal domain with maximum condition on principal right ideals. Therefore the skewfield \( Q(R) \) of quotients, \( Q(R) = \{ a^{-1}b, 0 \neq a, b \in R \} \), exists, but the subring \( T \) of \( Q(R) \) generated by \( R \) and \( x^{-1}tx \), \( T = R[x^{-1}tx] \), is not a ring of quotients of \( R \) with respect to some Ore-system of \( R \). To prove this last statement we observe first that \( R[x^{-1}tx] = \{ ax^{-1}dxb+c \) for \( a, b, c, d \in R \} \) since \( (x^{-1}tx)^2 = t \). If we assume \( T = S^{-1}R \) for some Ore-system \( S \) of \( R \), it follows that there exists an element \( f(x) \) in \( R \) of degree greater than 0, such that \( f(x)^{-1} \) is in \( T \). It is easy to prove that \( f(x) \) has to be equal to \( ux \) for a unit \( u \) in \( R \) and it would follow that \( x^{-1} \) is contained in \( T \). But this is not possible since from \( x^{-1} = ax^{-1}dxb+c, a, b, c, d \) in \( R \), it follows that \( 1 = a_1dxb+xc \) for some \( a_1 \) in \( R \), leading to a contradiction.

REFERENCES


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