PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

OVERRINGS OF PRINCIPAL IDEAL DOMAINS

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ABSTRACT. All rings between a (right and left) principal ideal domain R and its skewfield $Q(R)$ of quotients are quotient rings of R with respect to Ore-systems in R.

In [4] R. Gilmer and J. Ohm investigate commutative integral domains with the QR-property, i.e. with the property that every ring between the ring R and its quotient field $Q(R)$ is a ring of quotients of R with respect to some multiplicatively closed system in R. All Dedekind domains with torsion class group have the QR-property.

In this note we describe all overrings (i.e. rings between R and its skewfield $Q(R)$ of quotients) of a principal right and left ideal domain R, as left quotient rings with respect to appropriate Ore-systems in R. Contrary to the commutative case, we show that the left quotient ring with respect to an irreducible element need not be local. Also, a counterexample shows that for a principal left ideal domain which is not a principal right ideal domain, overrings exist which are not left quotient rings with respect to some Ore-system. See [1] for similar results.

Let R be a principal left and right ideal domain. Then R is a unique factorization domain, in which every nonunit element $0\neq a$ can be written as a product of irreducible elements:

$$a = p_1 \cdots p_n, \qquad p_i \text{ irreducible for all } i.$$

If $a=q_1 \cdots q_m$ is another irreducible factorization of a, then $n=m$ and there exists a permutation π of $\{1, \cdots, n\}$ such that R/Rp_i is isomorphic to $R/Rq_{\pi(i)}$ as an R-left module for every i ([5, p. 34]). We say an element a is similar to b in R, if R/Ra is isomorphic to R/Rb as a left-R module. It is well known, [3, p. 316], that a and b are similar if and only if there exists an element c in R with $Ra+Rc=R$ and $Ra\cap Rc=Rbc$. Similarity is an equivalence relation and left-right symmetric.

For each $a\neq 0$ in R and every irreducible element p in R we define a nonnegative integer $h_p(a)=n$, if n is the number of irreducible factors similar to p in any irreducible factorization of a. It follows

Received by the editors May 4, 1970.

AMS 1970 *subject classifications*. Primary 16A04; Secondary 16A08.

Key words and phrases. Overrings, principal ideal domains, local rings.

[1] This research was supported in part by the NRC of Canada, under Grant no. A-7254.

that $h_p(a)=0$ for all p if and only if a is a unit in R. For any set Δ of irreducible elements in R we define the set $S_\Delta=\{a\in R\ni h_p(a)=0$ for all p in $\Delta\}$.

We repeat the following definition: A multiplicatively closed set M of regular elements of R is called a (left-) Ore-system if for elements m in M and r in R there exist elements m_1 in M and r_1 in R such that $r_1m=m_1r$. If M is an Ore-system in R, then there exists the ring of quotients of R with respect to M ([6, p. 7]). With this notation we obtain the following result:

THEOREM. *Let R be a left and right principal ideal domain. Then any set S_Δ is an Ore-system and every ring between R and its skewfield of quotients $Q(R)$ can be obtained as a ring of quotients of R with respect to some S_Δ.*

PROOF. First we show that every S_Δ is an Ore-system. That S_Δ is multiplicatively closed follows from the unique factorization theorem stated in the beginning. It remains to show that for $s\in S_\Delta$ and r in R elements s_1 in S_Δ and r_1 in R exist with

$$s_1r = r_1s.$$

Let $Rs+Rr=Rd$, $s=s'd$, $r=r'd$, and it follows

$$Rs' + Rr' = R \qquad \text{for } s' \text{ in } S_\Delta \text{ and } r' \text{ in } R.$$

Further, $Rs'\cap Rr'=Rs_1r'$ and s_1 is a product of irreducible factors similar to the irreducible factors of s' and is therefore contained in S_Δ ([2, p. 51]). It follows that $s_1r'=r_1s'$ or $s_1r=r_1s$ for s_1 in S_Δ and r_1 in R. This proves that S_Δ is an Ore-system and there exists an overring $S_\Delta^{-1}R=\{s^{-1}a,\ s\in S,\ a\in R\}$ of R.

Now let T be any ring between R and $Q(R)$, $a^{-1}b$ an element in T. We may assume $bR+aR=R$, and from $bx+ay=1$, for x, y in R, it follows that a^{-1} belongs to T. Therefore, the inverse of every irreducible factor of a is contained in T. It remains to show that for p_1 similar to the irreducible element p in R with p^{-1} in T we have that p_1^{-1} is contained in T.

Since p_1 is similar to p, there exists c in R with $Rp\cap Rc=Rp_1c$ and $Rp+Rc=R$. The element cp^{-1} is contained in T and can be written in the form $cp^{-1}=a^{-1}b$ for $a\neq 0$, b in R. But then $ac=bp$ in $Rp\cap Rc$ and this means $ac=rp_1c$ or $a=rp_1$ for some element r in R. From this it follows that p_1^{-1} is contained in T. This proves that $T=S_\Delta^{-1}R$ where Δ is the set of those irreducible q in R such that q^{-1} is not contained in T.

REMARK 1. Unlike the commutative case, $S_p^{-1}R$ is in general not a local ring for Δ consisting of just one irreducible element p. ($S_p^{-1}R$ is local if and only if $h_p(a+b) \geqq \min(h_p(a), h_p(b))$ [**2**, p. 57].) For example, consider the irreducible element $p=x+t$ in the principal left and right ideal domain $R=\mathbf{Q}(t)[x, \sigma]=\{\sum_{i=0}^{n} f_i(t)x^i, f_i(t) \in \mathbf{Q}(t)\}$, where $\mathbf{Q}(t)$ is the field of rational functions in one variable over the field of rationals $\mathbf{Q}$ and σ is the automorphism of $Q(t)$ defined by $t^\sigma=t+1$. Addition in R is defined componentwise and multiplication by $xf(t)=f^\sigma(t)x$.

Now it is easy to see that $x+g(t)^{-1}\, g(t+1)(t+1)$ for every element $0 \neq g(t)$ in $\mathbf{Q}(t)$, is similar to the element $x+t$ in R. From this we conclude that for example $S_p^{-1}Rp$ and $S_p^{-1}Rp_1$, with $p_1=x+t^{-1}\,(t+1)^2$, are two different maximal left ideals of $S_p^{-1}R$ for $p=x+t$.

REMARK 2. The theorem just proved can not be extended to left principal ideal domains with maximum condition on right principal ideals, even though the rings $S^{-1}R$, as defined above, exist in this case too. Consider $\mathbf{Q}(t)[x, \tau]=R$ as in the previous remark, but τ is now the monomorphism from $\mathbf{Q}(t)$ into $\mathbf{Q}(t)$ defined by $t^\tau=t^2$. R is a principal left ideal domain with maximum condition on principal right ideals. Therefore the skewfield $Q(R)$ of quotients, $Q(R) = \{a^{-1}b, 0 \neq a, b \text{ in } R\}$, exists, but the subring T of $Q(R)$ generated by R and $x^{-1}tx$, $T=R[x^{-1}tx]$, is not a ring of quotients of R with respect to some Ore-system of R. To prove this last statement we observe first that $R[x^{-1}tx]=\{ax^{-1}dxb+c \text{ for } a, b, c, d \text{ in } R\}$ since $(x^{-1}tx)^2=t$. If we assume $T=S^{-1}R$ for some Ore-system S of R, it follows that there exists an element $f(x)$ in R of degree greater than 0, such that $f(x)^{-1}$ is in T. It is easy to prove that $f(x)$ has to be equal to ux for a unit u in R and it would follow that x^{-1} is contained in T. But this is not possible since from $x^{-1}=ax^{-1}dxb+c$, a, b, c, d in R, it follows that $1=a_1dxb+xc$ for some a_1 in R, leading to a contradiction.

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